Eilenberg on epimorphisms among groups

Group G, subgroup $H \subseteq G$. Can that inclusion be an epimorphism?

H of index 1 in G? — Yes ($H = G$).
H of index 2? — No (H is normal in G).
H of index ≥ 3? — Fix two cosets Ha, Hb of H, distinct from each other and from H, and define $\tau \in |G|!$ — an involution — by giving $\tau(x)$, for $x \in G$, as

$$\tau(x) = \begin{cases} xa^{-1}b \ (= h_0 b \in Hb), & \text{if } x = h_0 a \in Ha; \\ x, & \text{if } x \notin Ha \cup Hb; \\ xb^{-1}a \ (= h_0 a \in Ha), & \text{if } x = h_0 b \in Hb. \end{cases}$$

Write $\kappa_\tau\colon |G|! \to |G|!$ for conjugation by involution τ — $\kappa_\tau(\sigma) = \tau\cdot\sigma\cdot\tau$. Compare left-regular representation $\rho\colon G \to |G|!$ ($\{\rho(g)\}(x) = gx$) with the composition $\kappa_\tau\cdot\rho\colon G \to |G|! \to |G|!$:
For $h \in H$ and $x \in G$, $\{\{\kappa_\tau\cdot\rho\}(h)\}(x) = \{\kappa_\tau(\rho(h))\}(x) = \{\tau\cdot\rho(h)\cdot\tau\}(x) = \{\tau\cdot\rho(h)\}(\tau(x)) =$

$$= \begin{cases} \{\tau\cdot\rho(h)\}(\tau(h_0 a)) = \{\tau\cdot\rho(h)\}(h_0 b) = \tau(hh_0 b) = hh_0 a = \{\rho(h)\}(x), & \text{if } x = h_0 a \in Ha; \\ \{\tau\cdot\rho(h)\}(\tau(x)) = \{\tau\cdot\rho(h)\}(x) = \tau(hx) = hx = \{\rho(h)\}(x), & \text{if } x \notin Ha \cup Hb; \\ \{\tau\cdot\rho(h)\}(\tau(h_0 b)) = \{\tau\cdot\rho(h)\}(h_0 a) = \tau(hh_0 a) = hh_0 b = \{\rho(h)\}(x), & \text{if } x = h_0 b \in Hb. \end{cases}$$

So $\kappa_\tau\cdot\rho = \rho$ on H. But $\{\rho(a)\}(e) = ae = a$, while $\{\{\kappa_\tau\cdot\rho\}(a)\}(e) = \{\kappa_\tau(\rho(a))\}(e) = \{\tau\cdot\rho(a)\cdot\tau\}(e)$ $= \{\tau\cdot\rho(a)\}(\tau(e)) = \{\tau\cdot\rho(a)\}(e) = \tau(ae) = \tau(a) = b \neq a$, so $\kappa_\tau\cdot\rho \neq \rho$, and $H \subset G$ was not epi.

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Why treat index 2 as special case?
Why mix left regular representation of G with right cosets of H?
If you need room for a good involutory permutation τ, just make room — add another point ∞ to the space G/H of left cosets gH of H, forming $E = G/H \cup \{\infty\}$, have $\tau \in E!$ interchange H with ∞ but leave the rest of G/H alone. Then do as before:
Write $\kappa_\tau\colon E! \to E!$ for conjugation by the involution τ — $\kappa_\tau(\sigma) = \tau\cdot\sigma\cdot\tau$; compose left-regular representation $\rho\colon G \to (G/H)! \subset E!$ ($\{\rho(g)\}(xH) = gxH$, $\{\rho(g)\}(\infty) = \infty$) with $\kappa_\tau\colon E! \to E!$;
for $h \in H$ and $C \in G/H \cup \{\infty\}$ ($C = xH$ ($x \in G$) or $C = \infty$), calculate $\{\kappa_\tau\cdot\rho(h)\}(C) =$

$$\tau(\{\rho(h)\}(\tau(C))) = \begin{cases} \tau(\{\rho(h)\}(\tau(xH))) = \tau(\{\rho(h)\}(xH)) = \tau(hxH) = hxH = \rho(h)(C), & (C = xH \neq H); \\ \tau(\{\rho(h)\}(\tau(H))) = \tau(\{\rho(h)\}(\infty)) = \tau(\infty) = H = \{\rho(h)\}(C), & (C = H); \\ \tau(\{\rho(h)\}(\tau(\infty))) = \tau(\{\rho(h)\}(H)) = \tau(H) = \infty = \{\rho(h)\}(C), & (C = \infty), \end{cases}$$

$= \{\rho(h)\}(C)$, so that $\kappa_\tau\cdot\rho$ and ρ agree on H.
Now let $g \in G$, and suppose $\kappa_\tau\cdot\rho(g) = \rho(g)$. Then $\{\kappa_\tau\cdot\rho(g)\}(H) = \{\rho(g)\}(H) = gH$. But in fact $\{\kappa_\tau\cdot\rho(g)\}(H) = \tau(\{\rho(g)\}(\tau(H))) = \tau(\{\rho(g)\}(\infty)) = \tau(\infty) = H$. So $gH = H$, and $g \in H$.
Thus $H \subseteq G$ is the equalizer of $\kappa_\tau\cdot\rho$ and ρ. In fact, inclusion $\eta\colon (G/H)! \subset E!$ and composition $\kappa_\tau\cdot\eta\colon (G/H)! \subset E! \to E!$ have equalizer $\{\pi \in (G/H)! \mid \pi(H) = H\}$. Pf.: $\pi \in (G/H)!$, $\eta(\pi) = \{\kappa_\tau\cdot\eta\}(\pi)$ $\Rightarrow \pi(H) = \{\eta(\pi)\}(H) = \{\{\kappa_\tau\cdot\eta\}(\pi)\}(H) = \{\tau\cdot\eta(\pi)\cdot\tau\}(H) = \{\tau\cdot\eta(\pi)\}(\infty) = \tau(\infty) = H$, QED.

[(Joyal, Categories, 6/'12): $H \subseteq G$ as equalizer of $\kappa_\tau\cdot\rho$ and ρ is pullback of this along $\rho_o\colon G \to (G/H)!$]