THE FUNCTIONAL FOUNDATION OF MEASURE THEORY

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Chapter Zero

Categories and Functors

0.1 Introduction

The main goal of this paper was originally intended to be a proof of the Fubini theorem in the context of sigma measures on boolean sigma rings, without reference to measure spaces nor recourse, therefore, to the Stone space. Carathéodory [2] had already presented a development of measure theory without measure spaces in his posthumous book (which I had the opportunity to translate into English), but he omitted any discussion of a Fubini theorem. In the course of deciding where a product measure should live, the realisation developed that, while Carathéodory's guiding principles were sound, some of his definitions could be radically improved (and the interdependence of many constructions more easily demonstrated) by judicious use of ideas from the theory of categories and functors, especially the notions of adjointness, representability, and structure. When it became clear that the Fubini theorem depended on a hitherto undefined sigma tensor product of sigma rings, the need for a solid foundation in functor theory was confirmed. The result is that the major portion of this paper deals with functor-theoretic foundations, some of them previously unavailable, and only in the last chapter is a measure first mentioned.

The present chapter develops those aspects of the theory of categories and functors which are needed in the sequel. The main theorems are those of sections six, nine, and ten on the existence of functorial representations and on the transference of structures. The first four sections (§§0.2-0.5) expose, in definitions and examples, for the most part, the rudiments of categories and functors. The fundamental theorem on the existence of functorial representations is proved in §0.6, where some applications are indicated. Slomiński's results [26] on equationally definable classes of abstract algebras with infinitary operations are presented in §0.7, where it is proved that equational categories have direct sums and that equational functors have left adjoints. Injectives and projectives are discussed in a setting just broad enough to permit application to equational categories in §0.8. The construction of the pointed category canonically associated to a category with a sufficiently good left zero object (a productive one) occurs in §0.9, where is also defined (after Eilenberg) the category of structures over one category with values in another. It is proved there that there is a canonical transference functor from the category of costructures (definition dual to that of structure) over a category with productive left zero with values in a pointed category to the category of costructures over the associated pointed category. The category of structures with values in an equational category is reinterpreted, in §0.10, as a certain category of functors from a small category canonically associated to the given equational category, and a theorem on the transference of such equational structures is proved.

Chapter One applies the functor-theoretic foundations to various categories of lattices, notably boolean sigma rings and boundedly sigma-complete lattice-ordered vector spaces and algebras. Frequent use is made of the fact that equational functors have left adjoints. After basic definitions and their immediate consequences are presented, in the first two sections, sigma tensor products of sigma rings are defined in §1.3, and proved to exist; §1.4 contains the proof that tensoring is an exact functor. These results were announced in [19]. The problem of injectives and projectives in the category of sigma rings is taken up in §1.5, where several necessary conditions for the injectivity of sigma rings and for the existence of injective sigma rings are obtained. It is possible that these criteria can be used to prove the nonexistence of injective sigma rings. The machinery needed for analysis is set into motion in §1.6, where the universal definition for the module of step functions over a boolean ring with values or coefficients in a module appears. A number of results are proved tending to indicate that the module of step functions has as rich a structure as the module of coefficients. It is proved in §1.7 that the Borel functor (from the category of sets with Borel structure to the category of sigma rings) is sufficiently well behaved that the transference of equational structure theorem is applicable in a limited way.

As soon as the relevant (equational) categories of lattice-ordered vector spaces and algebras are defined in §1.8, the extraordinarily rich costructure on the sigma ring of Borel sets in the reals can be described; this description, using the transference theorem and the usual rich structure of the reals, and a comparison of the real-valued step functions on a sigma ring with the lattice-ordered algebra of sigma homomorphisms from the Borel sets of the reals are provided in §1.9. These results are of utmost importance in measure theory. Stone spaces were necessarily introduced in the proof that the Borel functor is reasonably well behaved. Their only reappearance is in §1.10, where they are used to prove that there is a largest boolean ring containing a given boolean ring as a dense ideal. The rôle they play in this connection is not crucial, however, as it is in connection with the Borel functor. Various properties and alternate descriptions of this largest dense extension are provided in the remainder of §1.10, for use later in the description of the dual of L_1.

With Chapter Two, we finally come to measure theory proper. The various kinds of (real-valued) finite measures are described in §2.1, where it is proved, as an immediate consequence of the definition of step functions, that the finite measures form the dual space of the (untopologised) space of real step functions, while the bounded measures form the continuous dual. §2.2 presents canonical projections from the space of bounded measures on a sigma ring to its subspace of countably additive measures, and from the space of countably additive measures on a complete boolean ring to its subspace of completely additive (<u>normal</u>) measures. In preparation for the fundamental theorem of calculus, the next section discusses measures that need not be finite and defines the supremum and infimum functions associated to a homomorphism from the Borel sets of the reals to a sigma ring. The fundamental theorem of calculus, incorporating the mean value inequalities, the Radon-Nikodym theorem, the chain rule, and a change of variables formula, occupies §2.4. The dual of L_1 is completely described in §2.5, using the material of §§2.4 and 1.10. The Fubini theorem, out of which this work arose, fittingly concludes it in §2.6.

There is quite a bit of literature on material related to the present work. Most prominent, of course, are the books of Carathéodory [2], Sikorski [25] (which contains an exhaustive bibliography), and Slomiński [26]. In [13], Haimo compares the Stone space of the direct or inverse limit of a family of unitary boolean rings with the inverse or direct limit of the Stone spaces. His result that the Stone space of a direct limit is the inverse limit of the Stone spaces was later very elegantly proved by Wallace [33], and the corresponding result in the other sense, which Haimo did not notice, that the Stone space of an inverse limit is the Stone-Cech compactification of the direct limit of the

Stone spaces, is proved quite easily in §1.7. The work of §2.2 is closely related to -- indeed, inspired by -- the work [10] of Gordon and Lorch. A consequence of our description of the dual of L_1 is half of the main theorem (Theorem 4) of Thorp's recent article [30], which was called to our attention some time after the program leading to this description was realised. The fundamental theorem of calculus, as presented here, is essentially due to Carathéodory [2]. Spaces of sigma homomorphisms from the Borel sets of the reals to a sigma ring play the crucial rôle in Kakutani's characteristation of abstract (L)-spaces, in Bohnenblust's characteristation of abstract L_p-spaces, and elsewhere (self-adjoint operators!), but they seem always to have been used, except by Goffman, as a heavy technical device to obtain a representation as functions on a topological space (the Stone space, in fact, of the boolean ring involved). It has been our aim, in contrast, to minimise the rôle of the Stone space, since its use tends to cloud some of the naturality of the proceedings.

A word should be said about the "category" of Banach spaces. The usual notion of a direct product (with sup norm) or of a direct sum (with L_1 norm) of Banach spaces fails to be the direct product or sum in the usual category-theoretic sense. Yet it is the case that, when the continuous linear transformations $\underline{\underline{BN}}(B, C)$ from one Banach space to another is viewed as a Banach space, there is a canonical isomorphism

$$\underline{\underline{EN}}(B, \mathop{\times}_{i\in I} B_i) \cong \mathop{\times}_{i\in I} \underline{\underline{EN}}(B, B_i) ,$$

where the product is in each instance the usual (sup norm) direct product of Banach spaces. This suggests that the definition of autonomous category given in (0.5.13) should be modifiable to take care of Banach spaces. Not having carried out this modification, we have thought it best to curtail the description of functoriality where Banach spaces are concerned, notably in Chapter Two. Fortunately, it is algebraic structures that are of predominant importance in the development of our theory, rather than questions of norm.

It has been necessary to omit consideration of other matters worthy of interest. Only the briefest mention is made of the fact that by use of the Borel sets of the complex numbers, complex measure theory can be developed in this context. The spaces L_p $(1 \neq p \neq \infty)$ and the phenomena associated with them are essentially neglected. The L_1 convolution theory arising from a sigma measure on a sigma ring equipped with the costructure of a group is totally absent, yet readily available with the material here presented. Finally, one should like to obtain a non commutative analogue of the whole theory, replacing boolean rings by a good category of (not necessarily distibutive) lattices. The fundamental problem, once the right class of lattices is selected, is to find one playing the same rôle there as the Borel sets of the reals do here.

The recent thesis of Holland at Harvard, which, I am informed, will soon be published, apparently initiates this theory with a Radon-Nikodym theorem that I have not seen.

It is my pleasant obligation to thank the National Science Foundation for four years of unstinting financial support, Drs. Michael Barr, Peter Freyd, and Haim Gaifman for several enlightening conversations, Mr. Aaron Galuten for the opportunity to translate [2], and Prof. E. R. Lorch for his constant interest and unfailing encouragement.

0.2 Categories

A <u>category</u> $\underline{\underline{A}}$ consists of the following data:

(0.2.1) a class of <u>indices</u> $A, B, \ldots$;

(0.2.2) a class of doubly indexed <u>sets</u> $\underline{\underline{A}}(A, B)$;

(0.2.3) a class of triply indexed <u>functions</u>

$$\varphi(\underline{\underline{A}})_{ABC} : \underline{\underline{A}}(A, B) \times \underline{\underline{A}}(B, C) \longrightarrow \underline{\underline{A}}(A, C) .$$

The indices $A, B, \ldots$ are called the <u>objects</u> of the category $\underline{\underline{A}}$; we may often write $A \in \underline{\underline{A}}$.

An element $f \in \underline{\underline{A}}(A, B)$ is called a <u>map</u> or <u>morphism</u> of the category $\underline{\underline{A}}$, or an $\underline{\underline{A}}$-morphism, more specifically, an $\underline{\underline{A}}$-morphism from A to B ; we may often write $f: A \longrightarrow B$ for $f \in \underline{\underline{A}}(A, B)$; if A and B are irrelevant, we may occasionally write simply $f \in \underline{\underline{A}}$.

The class of functions $\varphi(\underline{\underline{A}})_{ABC}$ is called the <u>composition rule</u> in A ; if $f: A \longrightarrow B$ and $g: B \longrightarrow C$, we generally write $g \cdot f$ or $g \circ f$ for $\varphi(\underline{\underline{A}})_{ABC}(f, g)$.

These data should satisfy the following axioms:

(0.2.4) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$;

(0.2.5) $\underline{\underline{A}}(A, B)$ is disjoint from $\underline{\underline{A}}(A', B')$ unless

$A = A'$ and $B = B'$;

(0.2.6) for each object A there is a map $e_A: A \longrightarrow A$ such that $f \cdot e_A = f$ and $e_A \cdot g = g$ for all $f: A \longrightarrow B$, all $g: B \longrightarrow A$, and all objects B .

(0.2.7) Lemma. If $\underline{\underline{A}}$ is a category the elements e_A are unique. Proof:

If there are two, say e_A and e_A' , then

$$e_A = e_A \cdot e_A' = e_A' .$$

The element e_A is called the identity map of A and will generally be written as id_A .

We proceed to give some examples of categories.

(0.2.8) $\underline{\underline{S}}$, the category of sets. An object of $\underline{\underline{S}}$ is a set; an $\underline{\underline{S}}$-morphism from A to B is just a function defined on the set A with values in the set B . The composition rule is the usual one: $(f \cdot g)(x) = f(g(x))$.

(0.2.9) $\underline{\underline{O}}$, the trivial category. This category has only one object and only one map (the composition rule is forced), the map automatically being the requisite identity map.

(0.2.10) The category associated to a partially ordered set $(P, \leq)$. The objects of this category are the elements of P. There is at most one map from one object A to another B, and the map is present iff $A \leq B$. The defining properties for a partial order ensure that the axioms of a category are fulfilled. Of especial interest will be the category, denoted $\downarrow$, associated to the set $\{0, 1\}$ equipped with the partial order $0 \leq 0 \leq 1 \leq 1$.

(0.2.11) If $\underline{\underline{A}}$ is a category, define a new category $\underline{\underline{A}}^*$ by specifying the objects of $\underline{\underline{A}}^*$ to be the objects of $\underline{\underline{A}}$, the maps $\underline{\underline{A}}^*(A, B)$ to be the maps $\underline{\underline{A}}(B, A)$, and the composition rule $\varphi(\underline{\underline{A}}^*)_{ABC}$ to be $\varphi(\underline{\underline{A}})_{CBA}$. Thus if the composition in $\underline{\underline{A}}$ is denoted by a dot, while that in $\underline{\underline{A}}^*$ is denoted by a star, the last specification becomes $g * f = f \cdot g$. The category $\underline{\underline{A}}^*$ is called the _dual_ category of $\underline{\underline{A}}$, and is very useful. The passage from $\underline{\underline{A}}$

to $\underline{\underline{A}}^*$ is reminiscent of the passage from a monoid to its opposite monoid, which has the same elements but the "opposite" multiplication. It is clear that $\underline{\underline{A}}^{**} = \underline{\underline{A}}$.

(0.2.12) If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are categories, define a new category $\underline{\underline{A}} \times \underline{\underline{B}}$ by specifying the objects to be the pairs (A, B) with $A \in \underline{\underline{A}}$ and $B \in \underline{\underline{B}}$. A map from (A, B) to (C, D) is by definition a pair of maps (f, g) with $f: A \longrightarrow C$ and $g: B \longrightarrow D$. Thus

$$(\underline{\underline{A}} \times \underline{\underline{B}})((A, B), (C, D)) = \underline{\underline{A}}(A, C) \times \underline{\underline{B}}(B, D) .$$

The composition rule is briefly given by the formula

$$(f, g) \cdot (f', g') = (f \cdot f', g \cdot g')$$

$\underline{\underline{A}} \times \underline{\underline{B}}$ is called the direct product of the categories $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

(0.2.13) If $\underline{\underline{A}}$ is a category, define a new category $\underline{\underline{M}} = \underline{\underline{Mor}}(\underline{\underline{A}})$ as follows. An object of $\underline{\underline{M}}$ is a map of $\underline{\underline{A}}$. If $f: A_0 \longrightarrow A_1$ and $g: B_0 \longrightarrow B_1$ are objects of $\underline{\underline{M}}$ (i.e., are $\underline{\underline{A}}$-morphisms), an $\underline{\underline{M}}$-morphism $\phi: f \longrightarrow g$ is by definition a pair of $\underline{\underline{A}}$-morphisms, $\phi = (\phi_0, \phi_1)$ with

$\phi_i \in \underline{\underline{A}}(A_i, B_i)$ $(i = 0, 1)$ of such a nature that the diagram of $\underline{\underline{A}}$-morphisms

$$\begin{array}{ccc} A_0 & \xrightarrow{f} & A_1 \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ B_0 & \xrightarrow{g} & B_1 \end{array}$$

commutes, i.e., that $g \cdot \phi_0 = \phi_1 \cdot f$. Finally, the composition may briefly be described by the formula

$$(\phi_0, \phi_1) \cdot (\psi_0, \psi_1) = (\phi_0 \cdot \psi_0, \phi_1 \cdot \psi_1) \ .$$

An $\underline{\underline{M}}$-morphism $\phi: f \longrightarrow g$ is also called a *transformation* from (the $\underline{\underline{A}}$-morphism) f to (the $\underline{\underline{A}}$-morphism) g .

(0.2.14) If $\underline{\underline{A}}$ is a category and C is a class of objects of $\underline{\underline{A}}$, the *full subcategory* of $\underline{\underline{A}}$ *generated* by C is the category whose objects are the objects in the class C and whose morphisms are all the $\underline{\underline{A}}$-morphisms between them.

0.3 Functors

A *functor* F from a category $\underline{\underline{A}}$ to a category $\underline{\underline{X}}$ consists of the following data:

(0.3.1) an $\underline{\underline{A}}$-indexed class of objects of $\underline{\underline{X}}$, i.e., an object $F(A) \in \underline{\underline{X}}$ for each $A \in \underline{\underline{A}}$;

(0.3.2) a doubly indexed class of functions

$$F_{AB}\colon \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{X}}(F(A), F(B)) .$$

The $\underline{\underline{X}}$-morphism $F_{AB}(f)$ (for $f \in \underline{\underline{A}}(A, B)$) is usually written simply as $F(f)$.

These data are subject to the following axioms:

(0.3.3) $F(id_A) = id_{F(A)}$;

(0.3.4) $F(g \cdot f) = F(g) \cdot F(f)$.

If datum (0.3.1) is accompanied not by (0.3.2) but by

(0.3.2*) a doubly indexed class of functions

$$F^{AB}\colon \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{X}}(F(B), F(A)) ,$$

and these data satisfy (0.3.3) and (instead of (0.3.4))

(0.3.4*) $F(g \cdot f) = F(f) \cdot F(g)$

(where again $F(f)$ means $F^{AB}(f)$), we call F a *contravariant functor* from $\underline{\underline{A}}$ to $\underline{\underline{X}}$.

(0.3.5) Lemma. If F is a (possibly contravariant) functor from $\underline{\underline{A}}$ to $\underline{\underline{B}}$, and G is a (possibly contravariant) functor from $\underline{\underline{B}}$ to $\underline{\underline{C}}$, then the data

$$G \cdot F\ (A) = G(F(A)) \quad , \quad G \cdot F\ (f) = G(F(f))$$

define a (possibly contravariant) functor from $\underline{\underline{A}}$ to $\underline{\underline{C}}$, called the composite of G with F and denoted $G \cdot F$. $G \cdot F$ is contravariant iff precisely one of F and G is. The identity function from $\underline{\underline{A}}$ to $\underline{\underline{A}}^*$ is a contravariant functor, composition with which converts every contravariant functor from (to) $\underline{\underline{A}}$ into a functor from (to) $\underline{\underline{A}}^*$ (and conversely, since $\underline{\underline{A}} = \underline{\underline{A}}^{**}$). Proof: immediate.

Next we present some examples of functors.

(0.3.6) The hom functor for a category $\underline{\underline{A}}$ is the functor, classically denoted $\mathrm{Hom}_{\underline{\underline{A}}} : \underline{\underline{A}}^* \times \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$, which is prescribed by the data

$$\mathrm{Hom}_{\underline{\underline{A}}}(A, B) = \underline{\underline{A}}(A, B) ,$$

$$\mathrm{Hom}_{\underline{\underline{A}}}(f, h)(g) = h \cdot g \cdot f$$

(i.e., for $f \in \underline{\underline{A}}^*(A, A') = \underline{\underline{A}}(A', A)$ and $h \in \underline{\underline{A}}(B, B')$,

$\mathrm{Hom}_{\underline{\underline{A}}}(f, h)$ is the function (i.e., S-morphism) from $\underline{\underline{A}}(A, B)$ to $\underline{\underline{A}}(A', B')$ which sends g (in $\underline{\underline{A}}(A, B)$) to $h \cdot g \cdot f$ (in $\underline{\underline{A}}(A', B')$). We also write $\mathrm{Hom}_{\underline{\underline{A}}} = \underline{\underline{A}}(\_,\_)$.

(0.3.7) The <u>partial hom functors</u> for a category $\underline{\underline{A}}$. Fix an object A in $\underline{\underline{A}}$. Define a functor $\underline{\underline{A}}_A : \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ by prescribing $\underline{\underline{A}}_A(B) = \underline{\underline{A}}(A, B)$ and $\underline{\underline{A}}_A(f) = \underline{\underline{A}}(id_A, f)$ (we also write $\underline{\underline{A}}(A, f)$ for $\underline{\underline{A}}(id_A, f)$). $\underline{\underline{A}}_A$, sometimes also written h_A , is called the <u>covariant</u> or <u>left hom functor defined by</u> A . The covariant hom functor $\underline{\underline{A}}^*_A : \underline{\underline{A}}^* \longrightarrow \underline{\underline{S}}$ is called the <u>contravariant</u> or <u>right hom functor defined by</u> A and is also denoted by $\underline{\underline{A}}^A$ or h^A when thought of as a contravariant functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$. It is specified by the data

$$h^A(B) = \underline{\underline{A}}^A(B) = \underline{\underline{A}}^*_A(B) = \underline{\underline{A}}^*(A, B) = \underline{\underline{A}}(B, A) ,$$

$$h^A(f) = \underline{\underline{A}}^A(f) = \underline{\underline{A}}^*_A(f) = \underline{\underline{A}}^*(A, f) = \underline{\underline{A}}(f, A) \underset{\mathrm{def}}{=} \underline{\underline{A}}(f, id_A).$$

(0.3.8) The functorial hom functors. To describe these functors, a general observation is in order: if F is a functor from $\underline{\underline{A}}$ to $\underline{\underline{X}}$ and G is a functor from $\underline{\underline{B}}$ to $\underline{\underline{Y}}$, there is a functor, denoted $F \times G$, from $\underline{\underline{A}} \times \underline{\underline{B}}$

to $\underline{\underline{X}} \times \underline{\underline{Y}}$, which is given by

$$(F \times G)(A, B) = (F(A), G(B)) ,$$

$$(F \times G)(f, g) = (F(f), G(g)) .$$

Now fix a functor $F: \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$. The <u>left functorial hom functor defined by</u> F is the functor

$$h_F: \underline{\underline{B}}^* \times \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$$

which is this composition:

$$\underline{\underline{B}}^* \times \underline{\underline{A}} \xrightarrow[F \times id_{\underline{\underline{A}}}]{} \underline{\underline{A}}^* \times \underline{\underline{A}} \xrightarrow[\underline{\underline{A}}(\_,\_)]{} \underline{\underline{S}} .$$

The <u>right functorial hom functor defined by</u> F is the functor

$$h^F: \underline{\underline{A}}^* \times \underline{\underline{B}} \longrightarrow \underline{\underline{S}}$$

which is this composition:

$$\underline{\underline{A}}^* \times \underline{\underline{B}} \xrightarrow[Id_{\underline{\underline{A}}^*} \times F]{} \underline{\underline{A}}^* \times \underline{\underline{A}} \xrightarrow[\underline{\underline{A}}(\_,\_)]{} \underline{\underline{S}} .$$

Similar definitions can be made for contravariant functors F .

(0.3.9) Natural transformations; the source and target functors. Form the category $\underline{\text{Mor}}(\underline{\underline{A}})$ of $\underline{\underline{A}}$-morphisms, as in (0.2.13). For $i = 0, 1$, the data

$$T_i(f: A_0 \longrightarrow A_1) = A_i ,$$

$$T_i((\phi_0, \phi_1)) = \phi_i$$

define two functors $T_i : \underline{\underline{\mathrm{Mor}}}(\underline{\underline{A}}) \longrightarrow \underline{\underline{A}}$ $(i = 0, 1)$ called, respectively, the source, or domain, functor and the target, or range, functor. Now let $T: \underline{\underline{B}} \longrightarrow \underline{\underline{\mathrm{Mor}}}(\underline{\underline{A}})$ be any functor. By composition, we get two functors

$$F_i = T_i \cdot T : \underline{\underline{B}} \longrightarrow \underline{\underline{A}} \qquad (i = 0, 1) ,$$

and T is called a <u>natural</u> <u>transformation</u> from F_0 to F_1 . Recalling the definition of the category $\downarrow$ (0.2.10), form the category $\underline{\underline{A}}^{\downarrow}$ whose <u>objects</u> are the functors from $\downarrow$ to $\underline{\underline{A}}$ and whose maps are the natural transformations between such functors. By assigning to each functor $F: \downarrow \longrightarrow \underline{\underline{A}}$ the $\underline{\underline{A}}$-morphism $F(0 \leq 1): F(0) \longrightarrow F(1)$ and to each natural transformation of such functors the corresponding transformation of $\underline{\underline{A}}$-morphisms, we obtain a complete identification of the category $\underline{\underline{\mathrm{Mor}}}(\underline{\underline{A}})$ with the category $\underline{\underline{A}}^{\downarrow}$. Finally, one can easily see that a natural transformation, e.g., a functor from $\underline{\underline{B}}$ to $\underline{\underline{\mathrm{Mor}}}(\underline{\underline{A}})$, being also a functor from $\underline{\underline{B}}$ to $\underline{\underline{A}}^{\downarrow}$, can equally well be thought of as a functor from $\underline{\underline{B}} \times \downarrow$ to $\underline{\underline{A}}$, and conversely.

0.4 Equivalences

(0.4.1) A map $f \in \underline{\underline{A}}(A, B)$ is called an *equivalence*, and the objects A and B are said to be *equivalent*, if there is a map $g \in \underline{\underline{A}}(B, A)$ such that

$$g \cdot f = id_A \qquad \text{and} \qquad f \cdot g = id_B \quad .$$

Such a map g is necessarily unique; one writes $g = f^{-1}$. A natural transformation $T: F_0 \longrightarrow F_1$ between two functors from $\underline{\underline{A}}$ to $\underline{\underline{B}}$ (i.e., a functor from $\underline{\underline{A}}$ to $\underline{\underline{Mor}}(\underline{\underline{B}})$) is called a (*natural*) *equivalence*, and the functors F_0 and F_1 are called (*naturally*) *equivalent*, if, for each object A in $\underline{\underline{A}}$, the $\underline{\underline{B}}$-morphism

$$T(A): F_0(A) \longrightarrow F_1(A)$$

is an equivalence (in this case the $\underline{\underline{A}}$-indexed class of $\underline{\underline{B}}$-morphisms $T(A)^{-1}$ defines a natural transformation denoted T^{-1} from F_1 to F_0 which is again, of course, a natural equivalence). A functor $F: \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is called an *equivalence*, and the categories $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are said to be *equivalent*, if there is a functor $G: \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$ such that $F \cdot G$ and $G \cdot F$ are naturally equivalent to the respective identity functors $id_{\underline{A}}$ and $id_{\underline{B}}$. If in fact there is a functor G for which $F \cdot G = id_{\underline{B}}$ and $G \cdot F = id_{\underline{A}}$, then F is called an <u>isomorphism</u> and we write $G = F^{-1}$. The G for an isomorphism F is unique; the G for an equivalence is unique to within unique natural equivalence.

One of the goals of functor theory is a characterisation of those functors which are naturally equivalent to one of the hom functors described in (0.3.6), (0.3.7), and (0.3.8). The first step in this direction is Yoneda's result [34] that there are no more natural transformations between h_A and h_B (cf. (0.3.7)) than there are maps between B and A, and that each natural equivalence between h_A and h_B arises from an equivalence between B and A. The presentation below, included for the sake of completeness, is adapted from lectures at Columbia by Albrecht Dold. Compare also Freyd [6, pp. 108-112].

If $T_i : \underline{A} \longrightarrow \underline{B}^{\downarrow}$ is a natural transformation from F_{i-1} to F_i $(i = 1, 2)$, define $T_2 \cdot T_1 : \underline{A} \longrightarrow \underline{B}^{\downarrow}$, a natural transformation from F_0 to F_2, by

$$T_2 \cdot T_1(A) = T_2(A) \cdot T_1(A)$$

$$T_2 \cdot T_1(f) = T_2(f) \cdot T_1(f) .$$

(Thus, if T is a natural equivalence from F to G, and T^{-1} is defined as in (0.4.1), $T^{-1} \cdot T$ is the "identity" natural transformation from F to F and $T \cdot T^{-1}$ is the "identity" transformation from G to G.) Take $\underline{\underline{B}} = \underline{\underline{S}}$ and let A and B be objects of $\underline{\underline{A}}$. We have already defined the functors h_A, $h_B : \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$; for $f \in \underline{\underline{A}}(A, B)$, we now define h_f, a natural transformation from h_B to h_A (note the inversion) by

$$h_f(C) = \underline{\underline{A}}(f, C) : h_B(C) \xrightarrow[=]{} \underline{\underline{A}}(B, C) \longrightarrow \underline{\underline{A}}(A, C) \xrightarrow[=]{} h_A(C)$$

$$h_f(g) = \underline{\underline{A}}(f, g) .$$

(0.4.2) Lemma. Let $F : \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ and $A \in \underline{\underline{A}}$. Then the class of all natural transformations from h_A to F forms a set $[h_A, F]$. The functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ which assigns to A the set $[h_A, F]$ and to $f : A \longrightarrow B$ the function $[h_f, F] : [h_A, F] \longrightarrow [h_B, F]$

defined by

$$[h_f , F](T) = T\cdot h_f$$

is naturally equivalent to F . <u>Proof</u>:

We content ourselves merely to show that there is a 1-1 correspondence between $[h_A , F]$ and $F(A) = FA$. This will prove the first assertion, and will leave little doubt concerning the second. Define a function

$$\Phi : FA \longrightarrow [h_A , F]$$

by defining, for each element a of FA , a natural transformation $\Phi(a): h_A \longrightarrow F$ as follows: for $X \in \underline{\underline{A}}$,

$$\Phi(a)(X): h_A(X) \longrightarrow FX$$

is the function sending f (an $\underline{\underline{A}}$-morphism from A to X) to $(\Phi(a)(X))(f) = F(f)(a)$ (i.e., evaluation); similarly, define a function $\Psi : [h_A , F] \longrightarrow FA$ by evaluation at at the identity, precisely, $\Psi(T) = T(A)(id_A)$ (recall that $T(A)$ is a function from $h_A(A) = A(A, A)$ to FA). We have immediately from the definitions of Φ and Ψ and from (0.3.3): if a is an element of FA, then

$$\Psi(\Phi(a)) = (\Phi(a))(A)(id_A) = F(id_A)(a) = id_{FA}(a) = a .$$

Having thus shown that $\Psi \cdot \Phi$ is the identity on FA, we proceed to show that $\Phi \cdot \Psi$ is the identity on $[h_A, F]$. It suffices to know that $\Phi \cdot \Psi(T) = T$ for each transformation $T: h_A \longrightarrow F$. For this, it is enough to verify that

(0.4.2') $\Phi(\Psi(T))(X)(f) = T(X)(f)$

holds for each $X \in \underline{\underline{A}}$ and each $f \in h_A(X) = \underline{\underline{A}}(A, X)$. And indeed, by the definitions of Φ and Ψ,

(0.4.2") $\Phi(\Psi(T))(X)(f) = \Phi(T(A)(id_A))(X)(f) = F(f)(T(A)(id_A))$,

and, since T is a natural transformation, the diagram

$$\begin{array}{ccc} h_A(A) & \xrightarrow{T(A)} & FA \\ {\scriptstyle h_A(f)} \downarrow & & \downarrow {\scriptstyle F(f)} \\ h_A(X) & \xrightarrow{T(X)} & FX \end{array}$$

commutes, whence the first equality below:

(0.4.2"') $F(f)(T(A)(id_A)) = T(X)(h_A(f)(id_A)) = T(X)(f)$

(the second equality is by definition (0.3.7) of $h_A(f)$). Combining (") and ("'), we get (') as desired.

Replacing $\underline{\underline{A}}$ by $\underline{\underline{A}}^*$, we obtain from (0.4.2):

(0.4.3) <u>Corollary</u>. For any functor $F: \underline{\underline{A}}^* \longrightarrow \underline{\underline{S}}$, there is a 1-1 correspondence $[h^B, F] \cong FB$ which is a natural equivalence of functors $\underline{\underline{A}}^* \longrightarrow \underline{\underline{S}}$.

Using (0.4.2) and (0.4.3) we obtain a result, an immediate consequence of which is the advertised Yoneda statement.

(0.4.4) Theorem. The three functors from $\underline{\underline{A}}^* \times \underline{\underline{A}}$ to $\underline{\underline{S}}$ which send (A, B) to

$$[h^A, h^B], \quad \underline{\underline{A}}(A, B), \quad [h_B, h_A]$$

respectively, are naturally equivalent among themselves.

Proof: simply combine all the natural equivalences and identifications at hand:

$$[h^A, h^B] \cong h^B(A) = \underline{\underline{A}}(A, B) = h_A(B) \cong [h_B, h_A].$$

(0.4.5) Corollary. The function $\underline{\underline{A}}(A, B) \longrightarrow [h_B, h_A]$ which sends f to h_f is a natural 1-1 correspondence under which equivalences correspond to natural equivalences.

Proof: It is clear, to begin with, that $h_f = \Phi(f)$. Naturality then follows from (0.4.4). The last assertion follows from naturality.

The discussion of functors naturally equivalent with hom functors continues in the next section.

0.5 *Representations and Adjoints*

(0.5.1) *Definition.* If the functor $F: \mathbf{A} \longrightarrow \mathbf{S}$ is naturally equivalent with a left hom functor h_A, it is *left representable*, and A *is a left representation for* F. If F is a contravariant functor from $\mathbf{A}$ to $\mathbf{S}$ and, viewed as a functor from $\mathbf{A}^*$ to $\mathbf{S}$, it is left representable by an object A, we obtain a natural equivalence between the contravariant functors F and h^A from $\mathbf{A}$ to $\mathbf{S}$: such a contravariant functor is called *right representable*, and the object A a *right representation for* F. If the functor $G: \mathbf{B}^* \times \mathbf{A} \longrightarrow \mathbf{S}$ is naturally equivalent with a left functorial hom functor h_F (for some $F: \mathbf{B} \longrightarrow \mathbf{A}$), G is called *left functorially representable* and F is a *left functorial representation for* G. If G is naturally equivalent with a right functorial hom functor h^E (for some $E: \mathbf{A} \longrightarrow \mathbf{B}$), we say G is *right functorially representable* and E is a *right functorial representation for* G. Finally, if G has both a left and a right functorial representation, say

F and E , respectively, then F is called a <u>left adjoint for</u> E and E a <u>right adjoint for</u> F .

The first thing to notice about this definition is that if a functor $F\colon \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ has two left representations, say A and B , composing the natural equivalences between h_A and F and between h_B and F yields a natural equivalence between h_A and h_B , which by (0.4.5) corresponds to an equivalence between A and B . The same argument applies to a contravariant functor, and so we have proved

(0.5.2) <u>Lemma</u>. Any two representations (left or right, as applicable) for a (possibly contravariant) functor are equivalent.

(0.5.3) <u>Definition</u>. Let $F\colon \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ and $A \in \underline{\underline{A}}$. An element $a \in FA$ that corresponds, by the Yoneda equivalence, to a natural equivalence from h_A to F is called <u>universal</u> for F . Clearly a functor is representable iff there is a universal element for it.

With this notion, we can refine (0.5.2) as follows.

(0.5.4) <u>Proposition</u>. If $a \in FA$ is universal for F and $a' \in FA'$ is arbitrary, there is a unique $\underline{\underline{A}}$-morphism $f: A \longrightarrow A'$ such that $F(f)(a) = a'$; moreover, the map f is an equivalence iff a' is also universal for F . <u>Proof</u>:

The composition $FA' \cong [h_{A'}, F] \cong [h_{A'}, h_A] \cong \underline{\underline{A}}(A, A')$ assigns the required f to a' . (0.4.5) finishes the proof.

In the next section we shall discuss functorial representations further. Here we present the most important examples of representations. For the rest of this section, we fix attention to functors from a category $\underline{\underline{A}}$ or its dual.

(0.5.5) Let $(A_i)_{i \in I}$ be a family of objects in $\underline{\underline{A}}$ indexed by a <u>set</u> I . A representation for the functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ which assigns to $B \in \underline{\underline{A}}$ the set

$$\bigtimes_{i \in I} \underline{\underline{A}}(A_i, B)$$

is called a <u>direct sum</u> of the family $(A_i)_{i \in I}$, and is denoted $\bigoplus_{i \in I} A_i$. The universal element, which is in

$\bigtimes_{i \in I} \underline{\underline{A}}(A_i, \bigoplus_{i \in I} A_i)$, is called the family of <u>canonical injections of the summands</u>.

(0.5.6) Again let $(A_i)_{i \in I}$ be a family of objects indexed by a set. A representation of the contravariant functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ which assigns to an object B the set $\bigtimes_{i \in I} \underline{\underline{A}}(B, A_i)$ is called a <u>direct product</u> of the A_i's and is denoted $\bigtimes_{i \in I} A_i$. The universal element in $\bigtimes_{i \in I} \underline{\underline{A}}(\bigtimes_{i \in I} A_i, A_i)$ is called the family of <u>canonical projections onto the factors</u>.

(0.5.7) Let $\underline{\underline{O}}$ be the zero category of (0.2.9), interpreted as a subcategory of $\underline{\underline{S}}$ by taking for the unique object a set consisting of a single point o. Let Z denote the only possible functor from $\underline{\underline{A}}$ to $\underline{\underline{O}}$. By default Z is both a functor and a contravariant functor. A left representation z_L for Z is called a <u>left zero object</u> of $\underline{\underline{A}}$, a right representation, z_R, a <u>right zero object</u> of $\underline{\underline{A}}$. By (0.4.4) there is exactly one map, say z_{LA} from z_L to an object A, and one map $z_{AR} : A \longrightarrow z_R$.

By their uniqueness, these maps must be compatible withh all $\underline{\underline{A}}$-morphisms, i.e., if $f \in \underline{\underline{A}}(A, B)$, then

$$f \cdot z_{LA} = z_{LB} \quad \text{and} \quad z_{BR} \cdot f = z_{AR} \ .$$

In particular, $z_{Lz_R} = z_{z_L R} : z_L \longrightarrow z_R$, and this map is an equivalence iff there is a map from z_Rt to z_L . In case z_L and z_R are equivalent, either is referred to simply as a <u>zero object</u> of $\underline{\underline{A}}$, and call $\underline{\underline{A}}$ a <u>category with zero</u>. Letting $\underline{\underline{S}}_*$ denote the category of sets with base points and base point preserving functions, a category $\underline{\underline{A}}$ is called a <u>pointed category</u> if, where $Ig: \underline{\underline{S}}_* \longrightarrow \underline{\underline{S}}$ is the functor which "ignores" the base point, a functor $\underline{\underline{h}}: \underline{\underline{A}}^* \times \underline{\underline{A}} \longrightarrow \underline{\underline{S}}_*$ can be found such that

$$\mathrm{Hom}_{\underline{\underline{A}}} = Ig \cdot \underline{\underline{h}}$$

and

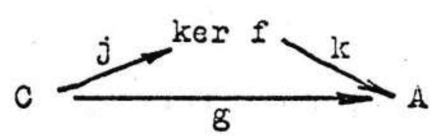
each $\varphi(\underline{\underline{A}})_{ABC}$ is bilinear.

Then <u>every category with zero is a pointed category</u>.

(0.5.8) Assume $\underline{\underline{A}}$ is pointed and let $f \in \underline{\underline{A}}(A, B)$. A representation of the functor from $\underline{\underline{A}}^*$ to $\underline{\underline{S}}$ which assigns to $C \in \underline{\underline{A}}$ the set

$$\ker(A(C, f)) = \left\{ g / \, g \in \underline{\underline{A}}(C, A), \ f \cdot g = 0 \right\}$$

(here O denotes the base point of each hom set) is called a <u>kernel</u> <u>for</u> f , and is denoted ker f . The universal element is an element $k \in \underline{\underline{A}}(\ker f, A)$ having the property that $f \cdot k = 0$ and whenever $f \cdot g = 0$ for $g \in \underline{\underline{A}}(C, A)$, $\exists^! \, j \in \underline{\underline{A}}(C, \ker f)$ such that the diagram



commutes.

This is a simple restatement of (0.5.4).

(0.5.9) Again assume $\underline{\underline{A}}$ is pointed and let $f \in \underline{\underline{A}}(A, B)$. A representation of the functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ which assigns to C in $\underline{\underline{A}}$ the set

$$\ker(\underline{\underline{A}}(f, C)) = \{ g / g \in \underline{\underline{A}}(B, C),\ g \cdot f = 0 \}$$

is called a <u>cokernel</u> <u>for</u> f and denoted cok f or coker f . The universal element has an interpretation similar to that given above for kernels (with arrows reversed, since the cokernel of and $\underline{\underline{A}}$-morphism is a kernel for the corresponding $\underline{\underline{A}}^*$-morphism).

In the next examples we will deal with categories of sets. A functor $F: \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ is <u>faithful</u> or <u>an</u>

immersion, if it is locally 1-1, that is to say, if each function $F_{AB}: \underline{\underline{A}}(A, B) \longrightarrow \underline{\underline{B}}(FA, FB)$ is 1-1. $\underline{\underline{A}}$ is said to have the structure of $\underline{\underline{B}}$ if there is an immersion from $\underline{\underline{A}}$ to $\underline{\underline{B}}$. $\underline{\underline{A}}$ is a concrete category, or a category of sets, if it has the structure of $\underline{\underline{S}}$. An immersion by virtue of which $\underline{\underline{A}}$ is concrete is usually held fixed throughout and denoted by "absolute value bars" or not at all; for A (resp. f) $\in \underline{\underline{A}}$, $|A|$ (resp. $|f|$) is thought of as the underlying set of A (resp. the underlying point function of f).

(0.5.10) Let $\underline{\underline{A}}$ be a category of sets and let S be a set. A representation for the functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ which assigns to an object A the set $\underline{\underline{S}}(S, |A|)$ is a free object generated by S. A free object generated by a singleton is the same as a left representation for the structural immersion $|\ |$, and is called a generator in $\underline{\underline{A}}$; more generally, any object G for which h_G is an immersion is a generator. If ΦS is a free object generated by S, the universal element, which lies in $\underline{\underline{S}}(S, |\Sigma S|)$, is called the <u>inclusion of the generators</u>; denoting it by k, one easily sees that to each $j \in \underline{\underline{S}}(S, |A|)$ there corresponds a unique $j' \in \underline{\underline{A}}(\Sigma S, A)$ such that $|j'| \cdot k = j \in \underline{\underline{S}}(S, |A|)$, which is essentially the usual "extension to a homomorphism" condition.

(0.5.11) Let $\underline{\underline{A}}$ be a concrete category, $B \in \underline{\underline{A}}$, and ρ an equivalence relation on $|B|$. Define a functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ by sending the object A to the set $\{f / f \in \underline{\underline{A}}(B, A),\ a \rho b \Longrightarrow |f|(a) = |f|(b)\}$. A representation of this functor is denoted B/ρ and is called a <u>quotient of B by (mod) ρ</u>; the universal element, in $\underline{\underline{A}}(B, B/\rho)$, is called the <u>canonical projection onto the quotient by ρ</u>, or <u>dividing by ρ</u>. In the case of a pointed concrete category, dividing by equivalence relations is closely allied to forming cokernels.

(0.5.12) Let $\underline{\underline{A}}$ be a category of sets and B an object of an arbitrary category $\underline{\underline{B}}$. An <u>$\underline{\underline{A}}$-structure on</u>

the object B is a contravariant functor $\mathbf{h}^B: \mathbf{B} \longrightarrow \mathbf{A}$ such that $|\mathbf{h}^B|$ (i.e., $|\ |\cdot\mathbf{h}^B$) and h^B are naturally equivalent. Dually, an $\mathbf{A}$-costructure on B is a functor $\mathbf{h}_B: \mathbf{B} \longrightarrow \mathbf{A}$ such that $|\mathbf{h}_B| \cong h_B$. For example, if $\mathbf{SG}$ denotes the category of monoids and monoid homomorphisms, to say that an object B in $\mathbf{B}$ has an $\mathbf{SG}$-structure involves, for each X in $\mathbf{B}$, a function $m(X): \mathbf{B}(X, B) \times \mathbf{B}(X, B) \longrightarrow \mathbf{B}(X, B)$, all of which form a natural transformation m; assuming that $B \times B$ exists in $\mathbf{B}$, this natural transformation is an element of $[h^{B\times B}, h^B]$, hence corresponds to an element of $\mathbf{B}(B \times B, B)$ called the *multiplication* on B. Similarly, an $\mathbf{SG}$-costructure on an object B corresponds, in case $B \oplus B$ exists in $\mathbf{B}$, to a *comultiplication*, that is, a $\mathbf{B}$-morphism $B \longrightarrow B \oplus B$. The situation will arise, in the main body of this work, of an object (in a concrete category) having the costructure of a lattice-ordered generalised Banach algebra. The number of structure maps is awkwardly large. The object is the Boolean ring of Borel sets on the real line. Other examples can also be concocted.

(0.5.13) A concrete category $\underline{\underline{A}}$ equippable with a functor $\underline{\underline{h}}: A^* \times A \longrightarrow A$ such that $|\underline{\underline{h}}| \cong \mathrm{Hom}_{\underline{\underline{A}}}$ is called <u>autonomous</u> provided the composition rule $\varphi(\underline{\underline{A}})_{ABC}$ can be lifted to a "bilinear $\underline{\underline{A}}$-morphism", i.e., is the underlying point function of some $\underline{\underline{A}}^*$-morphism $\underline{\underline{h}}(A, B) \longrightarrow \underline{\underline{h}}(\underline{\underline{h}}(B, C), \underline{\underline{h}}(A, C))$ <u>and</u> of some $\underline{\underline{A}}$-morphism $\underline{\underline{h}}(B, C) \longrightarrow \underline{\underline{h}}(\underline{\underline{h}}(A, B), \underline{\underline{h}}(A, C))$. Examples of autonomous categories are provided by $\underline{\underline{S}}$, $\underline{\underline{S}}_*$, $\underline{\underline{AG}}$ (the category of abelian groups and group homomorphisms), and $\underline{\underline{NV}}$ (the category of (say) real normed vector spaces and continuous homomorphisms). In an autonomous category, there can be defined a notion of tensor product. Namely, let A and B be two objects of an autonomous category $\underline{\underline{A}}$, and let $b: \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ be the functor sending X to

$$b(X) = \underline{\underline{A}}(A, \underline{\underline{h}}(B, X)) \cap \underline{\underline{A}}(B, \underline{\underline{h}}(A, X)),$$

the intersection occurring in $\underline{\underline{S}}(|A| \times |B|, |X|)$. A representation of b is called a <u>tensor product</u> of A with B and is denoted $A \otimes B$, or, when there is confusion as to which category is involved, $A \otimes_{\underline{\underline{A}}} B$. It should be observed that the natural isomorphisms (in $\underline{\underline{S}}$) between $|A| \times |B|$ and $|B| \times |A|$ and between $(|A| \times |B|) \times |C|$ and $|A| \times (|B| \times |C|)$ induce natural isomorphisms (in $\underline{\underline{A}}$)

$$A \otimes B \cong B \otimes A$$

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C) .$$

The first of these, when $B = A$, is called the <u>twisting automorphism</u> of $A \otimes A$; in view of the second, we shall never need parentheses when writing iterated tensor products. In each of the categories $\underline{\underline{S}}$, $\underline{\underline{S}}_*$, $\underline{\underline{AG}}$, $\underline{\underline{EN}}$, tensor products exist (for $\underline{\underline{EN}}$ this is due to Schatten [22] and Grothendieck [12, Chap. I, §1, n° 1, Prop. 1, p. 28]). The functor b assigns to X the socalled <u>bilinear maps</u> from $A \times B$, and the universal element has the usual universal property of "linearising" bilinear maps. Tensor products in other categories than autonomous ones all arise from a suitable notion of bilinearity or by comparison with a related autonomous category (see §1.3 for examples, and §1.6 for a tensor product like pairing of two categories to a third).

0.6 Functorial Representations

This section is devoted in the main to the proof of a proposition which gives rather easily verified criterion for the existence of a functorial representation, a criterion, moreover, which will frequently be used in the next chapter, often without explicit reference. The proposition in question is probably well known, but I have never seen a proof presented. Incidentally, application of this criterion to some of the examples of the previous section give examples of functorial representations.

(0.6.1) Proposition. If G is a functor from $\underline{\underline{B}} \times \underline{\underline{A}}$ to $\underline{\underline{S}}$, let $G(B): \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ (for each $B \in \underline{\underline{B}}$) denote the "partial" functor defined by

$$G(B)(A) = G(B, A) \quad , \quad G(B)(f) = G(id_B, f) .$$

Assume that each $G(B)$ is representable, say with representation F_B and universal element $i_B \in G(B)(F_B)$. Then F extends to a functor $\underline{\underline{B}}^* \longrightarrow \underline{\underline{A}}$, and the universal elements i_B define a natural equivalence of h_F with G, making F a functorial representation of G. Proof:

Proof that F is a functor. Let $\alpha \in \underline{\underline{B}}(B, B')$. Using (0.5.4), define $F_\alpha \in \underline{\underline{A}}(F_{B'}, F_B)$ to be the unique $\underline{\underline{A}}$-morphism for which the function

$$G(B')(F_\alpha): \ G(B')(F_{B'}) \longrightarrow G(B')(F_B)$$

sends $i_{B'}$ to the same element as the function

$$G(\alpha, id_{F_B}): G(B)(F_B) \longrightarrow G(B')(F_B)$$

sends i_B. That is, $f = F_\alpha$ is the unique solution f of the equation

$$G(\alpha, id_{F_B})(i_B) = G(B')(f)(i_{B'}).$$

In order to see that this definition makes F a functor, we must see that $F_{\beta\cdot\alpha} = F_\alpha \cdot F_\beta$ (contravariance!) for $\beta \in B(B', B'')$, and that $F_{id_B} = id_{F_B}$. The latter statement following immediately from the uniqueness part (0.5.4), it suffices here to prove the former. For that it is enough to prove

(0.6.2) $$G(\beta \cdot \alpha, id_{F_B})(i_B) = G(B'')(F_\alpha \cdot F_\beta)(i_{B''})$$

since this is the defining equation for $F_{\beta\cdot\alpha}$. Now since G is a functor, we have

$$(0.6.3)\qquad G(\beta\cdot\alpha\,,\ id_{F_B}) = (G(\beta\,,\ id_{F_B}))\cdot(G(\alpha\,,\ id_{F_B})\ .$$

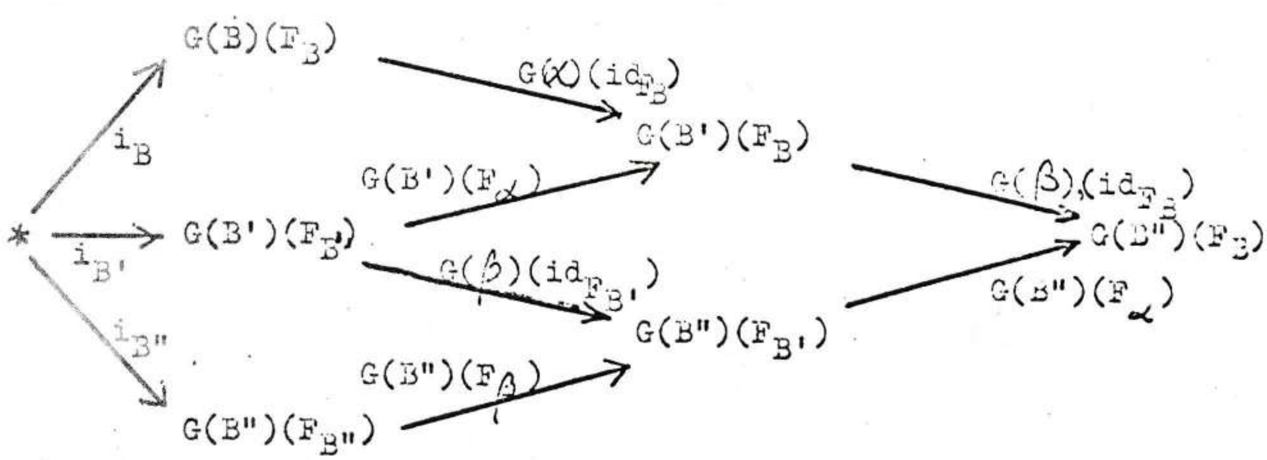
Similarly, since $G(B'')$ is a functor, we have

$$(0.6.4)\qquad G(B'')(F_\alpha\cdot F_\beta) = (G(B'')(F_\alpha))\cdot(G(B'')(F_\beta))\ .$$

Applying (0.6.3) and (0.6.4) to (0.6.2), we see that it will suffice to prove

$$(0.6.5)\qquad \begin{aligned}(G(\beta\,,\ id_{F_B}))\cdot(G(\alpha,\ id_{F_B}))(i_B) &= \\ &= (G(B'')(F_\alpha))\cdot(G(B'')(F_\beta))(i_{B''})\ .\end{aligned}$$

Accordingly, we make the convention that a point s of a set S is to be identified with (i.e., thought of as) a function (also written s) from a single point $*$ to S (sending $*$ to s). With this convention, consider the diagram



in which $G(\alpha)(id_{F_B})$, etc., is the same as $G(\alpha, id_{F_B})$, etc. Since G is a functor, $G(\beta)$ so defined is a natural transformation (from $G(B')$ to $G(B'')$); hence the right hand square commutes. The left hand squares each commute by definition of F_α and F_β, and so we have established (0.6.5) and proved F is a functor.

Proof that F is a functorial representation for G. It is convenient to designate the natural equivalence from $G(B)$ to h_{F_B} corresponding to the universal element $i_B \in G(B)(F_B)$ by the same symbol i_B, so that for $A \in \underline{\underline{A}}$, $i_B(A)$ is an equivalence $G(B)(A) \xrightarrow{\cong} h_{F_B}(A) = \underline{\underline{A}}(F_B, A)$. To see that the family $i_B(A)$ of equivalences sets up a natural equivalence of functors from $\underline{\underline{B}} \times \underline{\underline{A}}$, it is then enough to see it sets up a natural transformation, for which we must check that, when $\beta \in \underline{\underline{B}}(B, B')$, the diagram

(0.6.6)
$$\begin{array}{ccc} h_{F_B}(A) & \xleftarrow{\;i_B(A)\;} & G(B)(A) \\ \Big\downarrow {\scriptstyle h_{F_\beta}(A)} & & \Big\downarrow {\scriptstyle G(\beta)(A)} \\ h_{F_{B'}}(A) & \xleftarrow{\;i_{B'}(A)\;} & G(B')(A) \end{array}$$
,

where $h_{F_\beta}(A)$ sends $f \in h_{F_B}(A) = \underline{\underline{A}}(F_B , A)$ to $f \cdot F_\beta \in \underline{\underline{A}}(F_{B'} , A) = h_{F_{B'}}(A)$, commutes. So let $p \in G(B)(A)$. Following p clockwise around (0.6.6) gives an element $f' = (i_{B'}(A)) \cdot (G(\beta)(A))(p) \in h_{F_{B'}}(A) = \underline{\underline{A}}(F_{B'} , A)$ which is uniquely determined (using (0.5.4)) by the requirement

(0.6.7) $(G(B')(f'))(i_{B'}) = G(\beta)(A)(p)$.

Let f'' denote the element obtained by chasing p around (0.6.6) counterclockwise, i.e.,

$$f'' = (h_{F_\beta}(A)) \cdot (i_B(A))(p) = (i_B(A)(p)) \cdot F_\beta .$$

Now (0.6.6) commutes iff $f' = f''$, i.e., using (0.6.7), iff

(0.6.8) $(G(B')(f''))(i_{B'}) = G(\beta)(A)(p)$.

In order to prove (0.6.8), consider $f = i_B(A)(p)$, which is the first stop when chasing p around counterclockwise to get f'' , so that

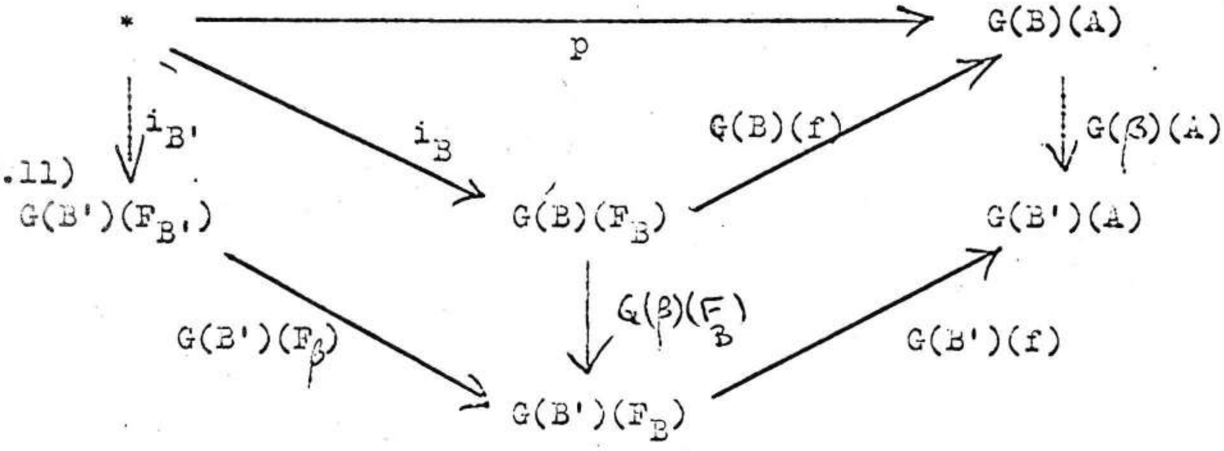
(0.6.9) $f'' = f \cdot F_\beta$.

This f is uniquely determined by the equation

(0.6.10) $p = (G(B)(f)(i_B)$.

Next, we remark that the diagram below commutes.

(0.6.11)



Indeed, the upper triangle commutes by (0.6.10), the left square commutes by definition of F_β , and the right square commutes since $G(\beta)$ is still a natural transformation from $G(B)$ to $G(B')$. Finally, application of the functor $G(B')$ to equation (0.6.9) gives the relation

$$(G(B')(f))\cdot(G(B')(F_\beta)) = G(B')(f'') ,$$

which, together with the commutativity of (0.6.11), yields the desired relation (0.6.8). This finishes the proof of Proposition (0.6.1).

Application of (0.6.1) to some of the examples (0.5.5)-(0.5.13) yields the following examples of functorial representations.

(0.6.12) Let A_2 be an object of $\underline{\underline{A}}$ and let $G_{A_2}: \underline{\underline{A}}^* \times \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ be the functor which assigns to the pair A_1, B the set $\underline{\underline{A}}(A_1, B) \times \underline{\underline{A}}(A_2, B)$. If each direct sum $A_1 \oplus A_2$ exists as an object of $\underline{\underline{A}}$, (0.6.1) assures that any choice of direct sum $A_1 \oplus A_2$ is part of a functor $\oplus A_2: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$, <u>summing with</u> A_2.

(0.6.13) Any choice of directiproduct $A_1 \times A_2$, for each A_1 and fixed A_2, is likewise part of a functor $\times A_2: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$, <u>multiplying with</u> A_2.

(0.6.14) Any choice of ker f , for each $\underline{\underline{A}}$-morphism f , is part of a functor ker: $\underline{\underline{A}}^{\downarrow} \longrightarrow \underline{\underline{A}}$. Similarly for coker .

(0.6.15) If in a <u>concrete</u> category $\underline{\underline{A}}$ a free object ΦS exists for each set S , any choice of free objects is part of a functor $\Phi: \underline{\underline{S}} \longrightarrow \underline{\underline{A}}$, which by definition (0.5.1) is a left adjoint to the immersion $|\,|: \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$.

(0.6.16) There is, if tensor products exist in an autonomous category $\underline{\underline{A}}$, a functor $\otimes A_2 : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}$, tensoring with A_2, which assigns to each object A_1 a tensor product $A_1 \otimes A_2$ of A_1 with A_2.

Just as representations are unique to within unique compatible equivalence, as follows from (0.5.2) and (0.5.4), so can this fact be used, in an argument like that proving (0.6.1), to show that functorial representations likewise are unique to within unique compatible natural equivalence. This comment is intended, in particular, for use with the functors obtained in the examples above.

One last comment: left functorial representations preserve left representations (the same is true replacing "left" by "right"), when they exist. This is an immediate consequence of the definition (0.5.1). Thus, for example, a functor which has a right adjoint preserves dirct sums, left zero objects, cokernels, and a functor which is a right adjoint preserves direct products, right zero objects, and kernels, among other things.

0.7 Equational Categories

Let A be a set, α an ordinal, and A^{α} the set of all sequences $(a_{\xi})_{0\leq\xi<\alpha}$ of elements of A. A function $F: A^{\alpha} \longrightarrow A$ is called an <u>operation</u> of <u>length</u> α on A, and A is called the <u>realm</u> of the operation F. If F and G are two operations of length α on realms A and B, respectively, a function $f: A \longrightarrow B$ is called a <u>homomorphism</u> from F to G if

$$f(F((a_{\xi})_{0\leq\xi<\alpha})) = G((f(a_{\xi}))_{0\leq\xi<\alpha}) .$$

Let $\Delta = (\alpha_{\lambda})_{0\leq\lambda<\beta}$ be a sequence of ordinals, and F_{λ} $(0\leq\lambda<\beta)$ an operation of length α_{λ} on A. The collection $(A; (F_{\lambda})_{0\leq\lambda<\beta})$ is called an <u>algebra</u> <u>of</u> <u>type</u> Δ, or a Δ-<u>algebra</u>. A Δ-<u>morphism</u> from a Δ-algebra $(A; (F_{\lambda}))$ to another $(B; (G_{\lambda}))$ is a function from A to B which is, for each available λ, a homomorphism from F_{λ} to G_{λ}.

The category of all Δ-algebras and Δ-morphisms is denoted (Δ), and is obviously concrete. A right zero object is afforded by the set consisting of a single element and the obvious (and unique) operations. Two theorems of Slomiński [26, Chap. III, (1.1) and (1.3)], when combined with (0.6.1), indicate that the immersion of $(\triangle)$ in $\underline{\underline{S}}$ by virtue of which $(\triangle)$ is concrete has a left adjoint, which, in accord with (0.5.10), is said to assign to the set S the <u>free</u> $\triangle$-<u>algebra</u> <u>on</u> (<u>generated</u> <u>by</u>) S. In particular, we have a free $\triangle$-algebra W on a set whose cardinality is the <u>rank</u> of $(\triangle)$, namely, whose cardinality is that of the least limit ordinal greater than or equal to each α_λ $(0 \leq \lambda < \beta)$.

A full subcategory $\underline{\underline{A}}$ of $(\triangle)$ is said to be <u>equationally</u> <u>defined</u> if there is a free $\triangle$-algebra W^* and a subset $E^* \subseteq |W^*| \times |W^*|$ such that a $\triangle$-algebra A belongs to $\underline{\underline{A}}$ if and only if each $\triangle$-morphism $f\colon W^* \longrightarrow A$ satisfies $f(x) = f(y)$ whenever $(x, y) \in E^*$, and E^* is called a <u>set</u> <u>of</u> <u>equations</u> <u>determining</u> $\underline{\underline{A}}$. An <u>equational</u> $\triangle$-<u>category</u> is an equationally defined (full) subcategory of $(\triangle)$. Slomiński has proved [26, Chap. III] that a category $\underline{\underline{A}}$ is an equational $\triangle$-category if and only if $\underline{\underline{A}}$ is determined by a set of equations in the free algebra W generated by a set of cardinality rank$(\triangle)$ (or greater), i. e., iff there is a subset $E \subseteq |W| \times |W|$ such that a $\triangle$-algebra A belongs to $\underline{\underline{A}}$ if and only if each $\triangle$-morphism $f\colon W \longrightarrow A$ satisfies $f(x) = f(y)$ whenever $(x, y) \in E$. This is a useful fact, a reinterpretation of which is given in §0.10. It is convenient to write $\underline{\underline{A}} = \triangle(E)$ if E is a set of equations in W determining the equational $\triangle$-category $\underline{\underline{A}}$.

Słomiński [26] proves the following results:

(0.7.1) (Δ) has direct products and quotients by relations of equivalence;

(0.7.2) If $\underline{\underline{A}}$ is an equational Δ-category, $f: B \longrightarrow A$ is a Δ-monomorphism, and $A \in \underline{\underline{A}}$, then $B \in \underline{\underline{A}}$;

(0.7.3) If $\underline{\underline{A}}$ is an equational Δ-category, $A \in \underline{\underline{A}}$, and ρ is an equivalence relation on $|A|$, then A/ρ, the quotient of A by ρ in (Δ), is in fact an object of $\underline{\underline{A}}$, and is the quotient of A by ρ in $\underline{\underline{A}}$;

(0.7.4) If $\underline{\underline{A}}$ is an equational Δ-category and A_i $(i \in I)$ are in $\underline{\underline{A}}$, then $\bigtimes_{i \in I} A_i$, the direct product in (Δ), is an object of $\underline{\underline{A}}$, and is the direct product in $\underline{\underline{A}}$ of the A_i's;

(0.7.5) The canonical "underlying set" immersion $\underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ from an equational Δ-category has an adjoint. (This again requires an application of (0.6.1) to Słomiński's result on the existence of free $\underline{\underline{A}}$-objects.)

We now prove

(0.7.6) Equational categories have direct sums.

Proof: Let $\underline{\underline{A}}$ be an equational Δ-category, and let $\Phi: \underline{\underline{S}} \longrightarrow \underline{\underline{A}}$ be the free object functor which is left adjoint to the canonical immersion of $\underline{\underline{A}}$ in $\underline{\underline{S}}$. Given A_i $(i \in I)$ objects of $\underline{\underline{A}}$, construct their direct sum as follows. First form $\Phi(\bigcup |A_i|)$ and $\Phi(|A_i|)$. There are, on the one hand, $\underline{\underline{A}}$-morphisms $\Phi(|A_i|) \xrightarrow[p_i]{} A_i$ (corresponding to $id_{|A_i|}$) and, on the other hand, $\underline{\underline{A}}$-morphisms $\Phi(|A_i|) \xrightarrow[j_i]{} \Phi(\bigcup|A_i|)$ (obtained by applying Φ to the inclusions $|A_i| \subseteq \bigcup|A_i|$). By the last comment of the previous section, it is clear that $(j_i)_{i \in I}$ is a universal element by virtue of which (cf. (0.5.5)) $\Phi(\bigcup|A_i|) = \bigoplus_{i \in I} \Phi(|A_i|)$. Also, each p_i induces an equivalence relation ρ_i on $|\Phi(|A_i|)|$ by the condition $a_i \, \rho_i \, b_i$ iff $p_i(a_i) = p_i(b_i)$. Now let ρ be the equivalence relation on $|\Phi(\bigcup|A_i|)|$ generated by transferring the ρ_i via $|j_i|$, that is to say, ρ is the equivalence relation on $|\Sigma(\bigcup |A_i|)|$ generated by the relation ρ_o defined as follows:

$$a \rho_o b \text{ iff } \exists i \in I \text{ and } a_i, b_i \in \Sigma(|A_i|)$$
$$\text{such that } a_i \rho_i b_i \text{ and } a = j_i(a_i), b = j_i(b_i) .$$

We claim: $\bigoplus_{i \in I} A_i = \Sigma(\bigcup |A_i|)/\rho$. Indeed, for B arbitrary in $\underline{\underline{A}}$, we have $\underline{\underline{A}}(\Sigma(\bigcup |A_i|)/\rho, B) \cong$ (def)

$$\cong \{f / f \in \underline{\underline{A}}(\Sigma(\bigcup |A_i|), B),\ a \rho b \Longrightarrow f(a) = f(b)\} \cong (\text{def } \rho)$$
$$\cong \{f / f \in \underline{\underline{A}}(\Sigma(\bigcup |A_i|), B),\ a_i \rho_i b_i \Longrightarrow f(j_i(a_i)) = f(j_i(b_i))\}$$
$$\cong \{(f_i)_{i \in I} / f_i \in \underline{\underline{A}}(\Sigma(|A_i|), B),\ a_i \rho_i b_i \Longrightarrow f_i(a_i) = f_i(b_i))\}$$
$$\cong \mathop{\times}_{i \in I} \underline{\underline{A}}(\Sigma(|A_i|)/\rho_i, B) \cong \mathop{\times}_{i \in I} \underline{\underline{A}}(A_i, B), \text{ where}$$

the last $\cong$ is due to the fact that $\Sigma(|A_i|)/\rho_i = A_i$ (cf. Slomiński [26, Chapter II, (5.8)]).

Remark: the same proof works for any concrete category with free objects, inwhich one can divide by relations, and where, if ρ is the relation on $|\Sigma(|A|)|$ determined by the canonical map $\Sigma(|A|) \longrightarrow A$, then $\Sigma(|A|)/\rho = A$.

Let $\Delta' = (\alpha_\lambda)_{0 \leq \lambda < \beta'}$, let $\beta \leq \beta'$, and put $\Delta = (\alpha_\lambda)_{0 \leq \lambda < \beta}$. Let W and W' be the free Δ-algebra and the free Δ'-algebra generated by a set of cardinality at least rank(Δ'). If E is a set of equations in W (i.e., $E \in |W| \times |W|$), denote by E also the resulting equations i(E) in W', image of E under the obvious Δ-morphism $i: W \longrightarrow W'$. Forming both $\Delta(E)$ and $\Delta'(E)$, there is an obvious functor from $\Delta'(E)$ to $\Delta(E)$, which simply ignores the operations from the β^{th} one on. Similarly, if $E \subseteq E' \subseteq |W| \times |W|$, we get an inclusion $\Delta(E') \longrightarrow \Delta(E)$. An <u>equational functor</u> is any composition of these two types of functors. The importance of equational functors lies in the following proposition.

(0.7.7) <u>Proposition</u>. Each equational functor is compatible with the standard immersions to $\underline{\underline{S}}$ and has a left adjoint. <u>Proof</u>:

The first part is obvious. Since the left adjoint of a composite is the composite, in the other order, of the individual left adjoints, when they exist, it suffices to prove the following lemma.

(0.7.8) <u>Lemma</u>. Each of these functors

$$\Delta'(E) \longrightarrow \Delta(E)\,, \quad \Delta(E') \longrightarrow \Delta(E)$$

has a left adjoint. <u>Proof</u>:

By (0.6.1) it will suffice to describe the action of the purported adjoint on each object. In the first instance, given a $\Delta(E)$-algebra A, form the free $\Delta'(E)$-algebra it generates, $\mathfrak{F}'(A)$. Define an equivalence relation on $\mathfrak{F}'(A)$ as follows: form $\mathfrak{F}(A)$, the free $\Delta(E)$-algebra generated by A, and let ρ be the equivalence relation on $\mathfrak{F}(A)$ determined by the canonical projection $\mathfrak{F}(A) \longrightarrow A$. Transfer ρ to $\mathfrak{F}'(A)$, that is to say, where $j\colon \mathfrak{F}(A) \longrightarrow \mathfrak{F}'(A)$ is the $\Delta(E)$-morphism induced by the inclusion of the generators $|A| \subseteq |\mathfrak{F}'(A)|$, write $a' \rho_0 b'$ (for a', b' in $\mathfrak{F}'(A)$) iff $a' = j(a)$ and $b' = j(b)$ with $a \rho b$. Let ρ' be the equivalence relation in $\mathfrak{F}'(A)$ generated by ρ_0; form $\mathfrak{F}'(A)/\rho'$. Now if B is a $\Delta'(E)$-algebra, a Δ-morphism from A to B extends to a Δ-morphism from $\mathfrak{F}(A)$ to B that identifies ρ-related elements; hence its extension to a Δ'-morphism from $\Phi'(A)$ to B identifies ρ'-related elements and is thus a $\Delta'(E)$-morphism from $\Phi'(A)/\rho'$ to B. Conversely, each such $\Delta'(E)$-morphism restricts to a $\Delta(E)$-morphism from $\Phi(A)$ to B identifying ρ-related elements, and so comes from a $\Delta(E)$-morphism from A to B. So the left adjoint to the first functor is described by specifying that A be snt to $\Phi'(A)/\rho'$.

In the second instance, given the $\Delta(E)$-algebra A, define an equivalence relation ρ directly on A as follows: for each Δ-morphism $f: W \longrightarrow A$, write $a \rho_f b$ if $a = f(x)$ and $b = f(y)$ with $(x, y) \in E'$, let ρ_o be the relation generated by all the ρ_f (i.e., $a \rho_o b$ iff for some f, $a \rho_f b$), and let ρ be the equivalence relation generated by ρ_o. Then A/ρ is a $\Delta(E')$-algebra; moreover, each $\Delta(E)$-morphism g from A to a $\Delta(E')$-algebra B has the property that if $a \rho_f b$ for some $f: W \longrightarrow A$, then $g(a) = g(b)$, since if

$a = f(x)$ and $b = f(y)$ with $(x, y) \in E$, then $g(a) = g \cdot f(x) = g \cdot f(y) = g(b)$. Hence also if $a \rho b$ then $g(a) = g(b)$. Thus every $\Delta(E)$-morphism from A to B identifies ρ-related elements, hence "is" a $\Delta(E')$-morphism from the $\Delta(E')$-algebra A/ρ to B. Thus the left adjoint to the second functor is described by specifying that A is sent to A/ρ.

<u>Remark</u>: This lemma will be applied early in the next chapter in the preliminary discussion on sigma rings.

0.8 Injectives and Projectives

(0.8.1) Definition. Let $F: \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ be an immersion (faithful functor). An element $f \in \underline{\underline{A}}(A, C)$ is called an F-monomorphism if $\underline{\underline{B}}(B, F(f)): B(B, FA) \longrightarrow \underline{\underline{B}}(B, FC)$ is 1-1 for all $B \in \underline{\underline{B}}$; it is an F-epimorphism if $\underline{\underline{B}}(F(f), B): \underline{\underline{B}}(FC, B) \longrightarrow \underline{\underline{B}}(FA, B)$ is 1-1 for all $B \in \underline{\underline{B}}$. An $id_{\underline{\underline{A}}}$-monomorphism (resp. $id_{\underline{\underline{A}}}$-epimorphism) is called simply a monomorphism (resp. epimorphism); if $\underline{\underline{B}} = \underline{\underline{S}}$, an F-monomorphism (resp. F-epimorphism) is described by the adjective 1-1 (resp. onto).

(0.8.2) Lemma. An F-monomorphism (resp. F-epimorphism) $f \in \underline{\underline{A}}(A, C)$ is also a monomorphism (resp. an epimorphism).

Proof: Let $g, h \in \underline{\underline{A}}(C, B)$, and suppose that f is an F-epimorphism (the proof for an F-monomorphism is dual) and that $g \cdot f = h \cdot f \in \underline{\underline{A}}(A, B)$. Then $F(g) \cdot F(f) = F(h) \cdot F(f)$ in $B(FA, FB)$, whence $F(h) = F(g)$ in $B(FC, FB)$, so that, since F is faithful, $h = g$ and $A(f, B)$ is 1-1, or f is an epimorphism.

(0.8.3) Definition. An object A is called a retract of C (by a pair of maps f, g) if $f \in \underline{\underline{A}}(A, C)$, $g \in A(C, A)$, and $g \cdot f = id_A$. Under the same circumstances, C is called an extension of A, and each of the maps f, g is said to be split by the other.

(0.8.4) Lemma. If A is a retract of C by a pair of maps f, g, then f is an F-monomorphism (resp. g is an F-epimorphism) for each immersion $F: \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$.

Proof: Since $g \cdot f = id_A$, $F(g) \cdot F(f) = id_{FA}$. Hence if $h, k \in B(B, FA)$ satisfy $F(f) \cdot h = F(f) \cdot k$, then

$h = id_{FA} \cdot h = F(g) \cdot F(f) \cdot h = F(g) \cdot F(f) \cdot k = id_{FA} \cdot k = k$,

so that $B(B, F(f))$ is 1-1 and f is an F-monomorphism. Likewise g is an F-epimorphism.

Remark: Such f (resp. g) is called a split monomorphism (resp. split epimorphism).

(0.8.5) Definition. An object P in $\underline{\underline{A}}$ is said to be F-projective (F an immersion from $\underline{\underline{A}}$) if

$$\underline{\underline{A}}(P, f): \underline{\underline{A}}(P, A) \longrightarrow \underline{\underline{A}}(P, C)$$

is onto for each F-epimorphism $f \in \mathbf{A}(A, C)$ and each pair of objects A, C. An object J in $\mathbf{A}$ is said to be F-<u>injective</u> if, for each F-monomorphism $f \in \mathbf{A}(A, C)$ and each pair of objects A, C, the function

$$\mathbf{A}(f, J): \mathbf{A}(C, J) \longrightarrow \mathbf{A}(A, J)$$

is onto. Finally, an object A in $\mathbf{A}$ is called an <u>absolute</u> F-<u>retract</u> (resp. <u>absolute</u> F-<u>coretract</u>) if every F-monomorphism $f: A \longrightarrow B$ (resp. every F-epimorphism $g: B \longrightarrow A$) splits, thus making A a retract of B. We abbreviate $id_{\mathbf{A}}$-projective, $id_{\mathbf{A}}$-injective, absolute $id_{\mathbf{A}}$-retract, absolute $id_{\mathbf{A}}$-coretract simply to projective, injective, absolute retract, absolute coretract; if $\mathbf{B} = \mathbf{S}$, we say $\mathbf{S}$-projective, etc.

As an immediate consequence of (0.8.2) and (0.8.5), we have the following proposition.

(0.8.6) <u>Proposition</u>. The following implications hold, for each immersion F from $\mathbf{A}$ to $\mathbf{B}$:

$$\begin{array}{ccc} \text{projective} & \Longrightarrow & \text{F-projective} \\ \Downarrow & & \Downarrow \\ \text{absolute coretract} & \Longrightarrow & \text{absolute F-coretract} \end{array} , \qquad \begin{array}{ccc} \text{injective} & \Longrightarrow & \text{F-injective} \\ \Downarrow & & \Downarrow \\ \text{absolute retract} & \Longrightarrow & \text{absolute F-retract} \end{array} .$$

The problem of describing the F-projectives (and the dual problem of the F-injectives) is solved, in a special case, as follows.

(0.8.7) Theorem. Suppose the immersion $F: \underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ has a left (resp. right) adjoint $\Phi: \underline{\underline{B}} \longrightarrow \underline{\underline{A}}$, and that every object of $\underline{\underline{B}}$ admits an epimorphism from a projective (resp. a monomorphism to an injective). Then an object A of $\underline{\underline{A}}$ having one of the three properties

i) A is F-projective

ii) A is a retract of $\Phi(Q)$ with Q projective in $\underline{\underline{B}}$

iii) A is an absolute F-coretract

(resp. i') A is F-injective

ii') A is a retract of $\Phi(J)$ with J injective in $\underline{\underline{B}}$

iii') A is an absolute F-retract)

has all three. Proof:

It is enough to prove the projective case, since the replacement of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ by $\underline{\underline{A}}^*$ and $\underline{\underline{B}}^*$ converts the injective case to the projective one. To this end, it clearly suffices to know these four facts:

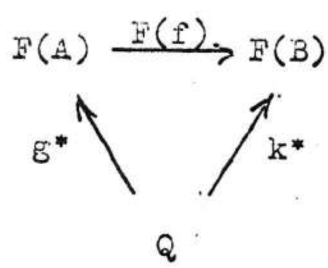
.1. $\Phi(Q)$ is F-projective if Q is projective;

.2. A retract of an F-projective is F-projective;

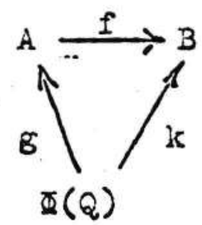
.3. Every F-projective is an absolute F-coretract;

.4. Every absolute F-coretract is a retract of some $\Phi(Q)$ with Q projective in $\underline{\underline{B}}$.

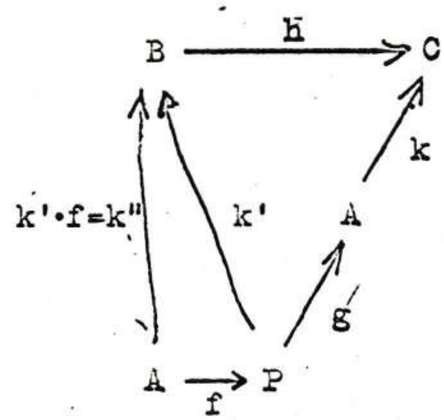
.1. Let $f \in \underline{\underline{A}}(A, B)$ be an F-epimorphism, and let $k: \Phi(Q) \longrightarrow B$. By adjointness ($\underline{\underline{A}}(\Phi(Q), B) \cong \underline{\underline{B}}(Q, F(B))$), k corresponds to a $\underline{\underline{B}}$-morphism $k^*: Q \longrightarrow F(B)$. Since Q is projective and $F(f)$ is an epimorphism, there is a $\underline{\underline{B}}$-morphism g^* such that the diagram



commutes; by adjointness g^* corresponds to an $\underline{\underline{A}}$-morphism $g: \Phi(Q) \longrightarrow A$ which, by the commutativity of the above diagram, makes the diagram below commute.



.2. Let A be a retract of the F-projective P by a pair of maps f , g ; let h: B ⟶ C be an F-epimorphism and let k: A ⟶ C . Find k'



such that $h \cdot k' = k \cdot g$, and put $k'' = k' \cdot f$. Then $h \cdot k'' = h \cdot k' \cdot f = k \cdot g \cdot f = k \cdot id_A = k$.

.3. follows from (0.8.6).

.4. Let P be an absolute F-coretract. Let $f: Q \longrightarrow F(P)$ be an epimorphism in $\underline{\underline{B}}$ with Q projective. By adjointness, f corresponds to an $\underline{\underline{A}}$-morphism $g: \Phi(Q) \longrightarrow P$. P , being an absolute F-coretract, will be a retract of $\Phi(Q)$, as desired, if g is an F-epimorphism, i.e., if $F(g): F(\Phi(Q)) \longrightarrow F(P)$ is an epimorphism in $\underline{\underline{B}}$. But if $u: Q \longrightarrow F(\Phi(Q))$ is the universal element, then by (0.5.4) $F(g)\cdot u = f$; since f is an epimorphism, $F(g)$ is an epimorphism, too. This completes the proof.

Recall that an object G in $\underline{\underline{A}}$ has been called a generator if $\underline{\underline{A}}_G: \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ is an immersion. Dually, a generator K of $\underline{\underline{A}}^*$ is called a <u>cogenerator</u> of $\underline{\underline{A}}$; so K is a cogenerator of $\underline{\underline{A}}$ iff $\underline{\underline{A}}^K: \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ is a <u>contravariant</u> immersion.

(0.8.8) <u>Lemma</u>. If $\underline{\underline{A}}$ has a generator G and $|\ | = \underline{\underline{A}}_G: \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$, then the monomorphisms and the $\underline{\underline{A}}_G$-monomorphisms coincide, whence the injectives and the $\underline{\underline{A}}_G$-injectives are the same. If $\underline{\underline{A}}_G$ has a left adjoint Φ (read "free", as usual), then the following conditions are equivalent:

i) G is projective

ii) the projectives coincide with the $\underline{\underline{A}}_G$-projectives

iii) every epimorphism is onto. <u>Proof</u>:

To verify the first assertion, let $f \in \underline{\underline{A}}(A, C)$ be a monomorphism, and let $g, h \in \underline{\underline{S}}(S, A) = \underline{\underline{S}}(S, \underline{\underline{A}}(G, A))$. Assuming that $|f| \cdot g = |f| \cdot h$ in $\underline{\underline{S}}(S, \underline{\underline{A}}(G, C))$, we have, for each point $s \in S$,

$$f \cdot (g(s)) = |f|(g(s)) = |f|(h(s)) = f \cdot (h(s))$$

in $\underline{\underline{A}}(G, C)$, whence, since f is a monomorphism, $g(s) = h(s)$ for all s; this shows $g = h$ and f is an $\underline{\underline{A}}_G$-monomorphism. (0.8.2) completes the proof.

For the second assertion, $G = \Sigma(\text{point})$, hence by (0.8.7) G is $\underline{\underline{A}}_G$-projective, and so ii) $\Longrightarrow$ i). That iii) $\Longrightarrow$ ii) is immediate. Finally i) $\Longrightarrow$ iii) since if G is projective and f is an epimorphism, then $\underline{\underline{A}}(G, f)$ is onto; but $\underline{\underline{A}}(G, f) = \underline{\underline{A}}_G(f) = |f|$. This completes the proof.

We shall deal, in the next chapter, with a situation in which the existence of one nontrivial injective implies i).

(0.8.9) Remark: It folows from (0.8.7), (0.7.5), and the fact that every object of $\underline{\underline{S}}$ is projective, that the $\underline{\underline{S}}$-projectives, the absolute $\underline{\underline{S}}$-coretracts, and the retracts of free objects in an equational category all coincide.

0.9 Associated pointed categories

In this section, we discuss a phenomenon which can most easily by described, somewhat inaccurately, by the illegitimate statement that the inclusion functor from the category of categories with zero to the category of categories with left zero has a left adjoint. More accurately, to each category with a sufficiently good left zero is associated a pointed category satisfying certain universal conditions. These conditions will allow a tidy description of the transference to the associated pointed category of a costructure on the original. Such a description is needed in §1.9.

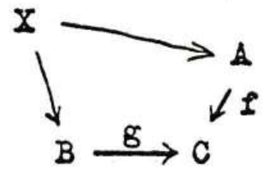
(0.9.1) Definition. Let A, B, C be three objects of a category $\underline{\underline{B}}$, and let $f\colon A \longrightarrow C$, $g\colon B \longrightarrow C$ be $\underline{\underline{B}}$-morphisms. The diagram $A \xrightarrow{f} C \xleftarrow{g} B$ is a pullback diagram; a pullback of this pullback diagram is a right representation of the contravariant functor $P_{f,g}\colon \underline{\underline{B}} \longrightarrow \underline{\underline{S}}$ defined (on objects) by

$$P_{f,g}(X) = \{(h,k) \,/\, h \in \underline{\underline{B}}(X,A),\ k \in \underline{\underline{B}}(X,B),\ f\cdot h = g\cdot k\} .$$

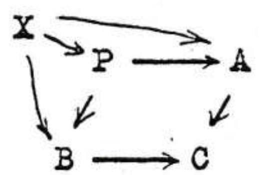
Thus if P is a pullback of the pullback diagram $A \xrightarrow{f} C \xleftarrow{g} B$, the universal element in $P_{f,g}(P)$ is a pair of maps from P, one to A, one to B, making a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & A \\ \swarrow & & \swarrow \\ B & \longrightarrow & C \end{array}$$

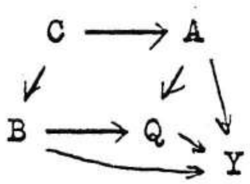
, such that whenever any diagram



commutes, there is a <u>unique</u> map from X to P making the diagram



commute. A <u>pushout</u> <u>diagram</u> is a pair of maps, which, when transported to the dual category, becomes a pullback diagram there; a <u>pushout</u> of a pushout diagram is an object which, when everything is transported to the dual category, is a pullback of the pullback diagram there. In the original category, the situation, schematically depicted, is



.

An <u>equaliser</u> of two maps f, $g \in \underline{\underline{B}}(A, C)$ is just a pullback of the pullback diagram $A \xrightarrow{f} C \xleftarrow{g} A$; coequalisers are defined dually. A <u>difference</u> <u>kernel</u> of two maps $f, g \in \underline{\underline{B}}(A, C)$ is a right representation of the contravariant functor $DK_{f,g}\colon \underline{\underline{B}} \longrightarrow \underset{\sim}{S}$ defined (on objects) by

$$DK_{f,g}(X) = \{ h / h \in B(X, A),\ f\cdot h = g\cdot h \},$$

i.e., is an object equipped with a map -- the universal element -- to A through which every map h to A satisfying $f \cdot h = g \cdot h$ factors uniquely. Difference cokernels are defined dually. It is not misleading to use the suggestive notation $\ker(f-g)$ for the difference kernel of f and g.

(0.9.2) Lemma. Consider the following assertions about a category $\underline{\underline{B}}$:

1) every pair of objects has a direct product
2) all equalisers exist
3) all pullbacks exist
4) all difference kernels exist
5) there is a right zero
6) $\underline{\underline{B}}$ is pointed
7) all kernels exist ;

then the following implications are valid:

$$3) \Longrightarrow 2) \qquad 1) \Longrightarrow [2) \Longrightarrow 4)]$$

$$1) \Longrightarrow [4) \Longrightarrow 3)] \qquad 5) \Longrightarrow [2) \Longrightarrow 1)]$$

$$6) \Longrightarrow [4) \Longrightarrow 7)] \qquad 6) \Longrightarrow [2) \Longrightarrow 1)].$$

Proof: The first implication is obvious. To construct $\ker(f-g)$, use the equaliser of the two maps $A \longrightarrow A \times B$, corresponding to the pairs of maps $(id_A, f:A \longrightarrow B)$ and $(id_A, g:A \longrightarrow B)$. To construct the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$, use the difference kernel of the pair of maps

$$A \times B \xrightarrow{proj.} A \xrightarrow{f} C$$

$$A \times B \xrightarrow{proj.} B \xrightarrow{g} C .$$

To construct the product of A with B, use the pullback of the diagram $A \longrightarrow z_R \longleftarrow B$ in case there is a right zero z_R, and of the diagram $A \longrightarrow C \longleftarrow B$ with any object C, but using the zero maps, in case the category is pointed. Finally, if the category is pointed, $\ker(f) = \ker(f - 0)$, where 0 is the zero map from the domain of f to the range of f.

Remarks: The constructions in the proof indicate that a functor preserving certain of these notions preserves all other notions implied by them. Notice that a functor having a left adjoint preserves all the notions in the theorem, whenever they exist, since they are right representations. A dual theorem is of course available for direct sums, coequalisers, pushouts, difference cokernels, left zeros, and cokernels; and a functor having a right adjoint preserves all these dual notions.

If $\underline{\underline{B}}$ is any category, define $\underline{\underline{B}}^!$ to be the full subcategory of $\underline{\underline{B}}^{\downarrow} = \underline{\underline{Mor}}(\underline{\underline{B}})$ whose objects are all $\underline{\underline{B}}$-morphisms to a left zero of $\underline{\underline{B}}$. If $\underline{\underline{B}}$ has no left zero, of course, $\underline{\underline{B}}^!$ is void, but if $\underline{\underline{B}}$ has a left zero, say z_L, then id_{z_L} is a (two-sided) zero object in $\underline{\underline{B}}^!$. The category of sets, whose left zero is the empty set, is an example of a category for which id_{z_L} is the only object of $\underline{\underline{B}}^!$, but such pathological cases need not be distinguished and cast out of the theory. The source functor $T_0: \underline{\underline{Mor}}(\underline{\underline{B}}) \longrightarrow \underline{\underline{B}}$ of (0.3.9) restricts to to a functor from $\underline{\underline{B}}^{!}$ to $\underline{\underline{B}}$, which we still denote as T_o; similarly, the restriction to $\underline{\underline{B}}^{!}$ of the target functor $T_1: \underline{\underline{B}}^{\vee} \longrightarrow \underline{\underline{B}}$ will still be called T_1. Thus, an object $m \in \underline{\underline{B}}^{!}$ is a $\underline{\underline{B}}$-morphism $m: T_o(m) \longrightarrow T_1(m) = z_L$, and a $\underline{\underline{B}}^{!}$-morphism $f: m \longrightarrow n$ "is" a $\underline{\underline{B}}$-morphism $T_o(f): T_o(m) \longrightarrow T_o(n)$ satisfying $m = n \cdot T_o(f)$.

(0.9.3) Lemma. Assume that $\underline{\underline{B}}$ has a left zero z_L and that each object of $\underline{\underline{B}}$ has a direct product with z_L. Then the functor $p: \underline{\underline{B}} \longrightarrow \underline{\underline{B}}^{!}$ which assigns to each object B the canonical projection from $B \times z_L$ to z_L has T_o as its left adjoint. Moreover, if $m \in \underline{\underline{B}}(B, z_L)$, then $p(m): p(B) \longrightarrow p(z_L)$ has a kernel in the category $\underline{\underline{B}}^{!}$ -- namely, the $\underline{\underline{B}}^{!}$-object m.

Proof: That p is a functor to begin with is due to (0.6.13) and (0.6.1). To produce an equivalence

$$\underline{\underline{B}}(T_o(m), B) \cong \underline{\underline{B}}^{!}(m, p(B)),$$

assign to the $\underline{\underline{B}}^{!}$-morphism $f: m \longrightarrow p(B)$ the composite of the $\underline{\underline{B}}$-morphism $T_o(f): T_o(m) \longrightarrow T_o(p(B)) = B \times z_L$ with the projection of $B \times z_L$ to B, and to the $\underline{\underline{B}}$-morphism $g: T_o(m) \longrightarrow B$ the $\underline{\underline{B}}^{!}$-morphism from m to $p(B)$ which sends $T_o(m)$ to $T_o(p(B)) = B \times z_L$ by the $\underline{\underline{B}}$-morphism corresponding to the pair (g, m). The universal element making $m: B \longrightarrow z_L$ the kernel of the $\underline{\underline{B}}^{!}$-morphism $p(m): p(B) \Big\downarrow^{B \times z_L}_{z_L} \longrightarrow p(z_L) \Big\downarrow^{z_L \times z_L}_{z_L}$ is the

$\underline{\underline{B}}^!$-morphism from $m\Big\downarrow\begin{matrix}B\\ \\ z_L\end{matrix}$ to $p(B)\Big\downarrow\begin{matrix}B\times z_L\\ \\ z_L\end{matrix}$ which sends B to B to $B\times z_L$ by the $\underline{\underline{B}}$-morphism corresponding to the pair of maps $(id_B\, ,\ m)$.

It will be convenient to speak of a category satisfying the hypothesis of Lemma (0.9.3) as having a productive left zero. The coming lemma on categories with productive left zeros is crucial.

(0.9.4) Lemma. Let $\underline{\underline{A}}$ be a pointed category, let $\underline{\underline{B}}$ be a category with a productive left zero, and let F be a functor from $\underline{\underline{B}}$ to $\underline{\underline{A}}$. Assume that whenever $m \in \underline{\underline{B}}(B, z_L)$ the $\underline{\underline{A}}$-morphism $F(m)\colon F(B) \longrightarrow F(z_L)$ has a kernel, and define $F^!\colon \underline{\underline{B}}^! \longrightarrow \underline{\underline{A}}$ by $F^!(m) = \ker(F(m))$.

.1. If F preserves products with z_L, i.e., if $F(B\times z_L) \cong F(B)\times F(z_L)$, then $F^!\cdot p \cong F$.

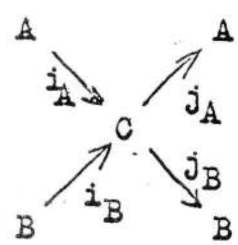
.2. If, in addition to preserving products with z_L, F also preserves the pullbacks, when they exist, of all pullback diagrams of the form $A \longrightarrow C \longleftarrow z_L$, then $F^!$ preserves kernels, when they exist, and any kernel-preserving functor whose composite with p is F must be naturally equivalent to $F^!$.

.3. Parts .1. and .2. above and the conclusions of Lemma (0.9.3) characterise the category $\underline{\underline{B}}^!$ to within unique equivalence.

Proof: We begin by disposing of the uniqueness of $\underline{\underline{B}}^!$. If $\underline{\underline{B}}^!$ and $\underline{\underline{B}}^1$ are two categories (pointed) equipped with functors $p\colon \underline{\underline{B}} \longrightarrow \underline{\underline{B}}^{!}$, $p^{1}\colon \underline{\underline{B}} \longrightarrow \underline{\underline{B}}^{1}$, each of which has a left adjoint, and for which $p(m)$ has a kernel in $\underline{\underline{B}}^{!}$ and $p^{1}(m)$ has a kernel in $\underline{\underline{B}}^{1}$ for each $\underline{\underline{B}}$-morphism m with target z_L (these are the conclusions of (0.9.3)), then both p and p^{1} preserve all products and pullbacks, being right adjoints, and so, by .1. and .2., there are unique kernel-preserving functors $u\colon \underline{\underline{B}}^{!} \longrightarrow \underline{\underline{B}}^{1}$, $v\colon B^{1} \longrightarrow B^{!}$, satisfying $u\cdot p \cong p^{1}$, $v\cdot p^{1} \cong p$. But $u\cdot v$ (resp. $v\cdot u$), which is kernel preserving since u and v are, satisfies $u\cdot v\cdot p^{1} \cong u\cdot p \cong p^{1}$ (resp. $v\cdot u\cdot p \cong v\cdot p^{1} \cong p$), whence, by .2., $u\cdot v \cong id_{\underline{\underline{B}}^{1}}$ (resp. $v\cdot u \cong id_{\underline{\underline{B}}^{!}}$). This proves .3. .

Part .1. is an immediate consequence of the following well known characterisation of direct products in pointed categories.

(0.9.5) Lemma. In a pointed category $\underline{\underline{A}}$ let four maps



be given. Then the two sets of conditions

.1. a) j_A and j_B make $C = A \times B$

b) i_A (resp. i_B) corresponds to the pair of maps $(id_A, 0)$ (resp. $(0, id_B)$)

and .2. a) $j_A \cdot i_A = id_A$ and $j_B \cdot i_B = id_B$

b) i_A makes $A = \ker(j_B)$ and i_B makes $B = \ker(j_A)$

are mutually equivalent.

This lemma is too well known to be proved here. It establishes .1. because $F^{!}(p(B)) = \ker(F(B \times z_L) \xrightarrow{F(\text{proj.})} F(z_L)) = \ker(F(B) \times F(z_L) \xrightarrow{\text{proj.}} F(z_L)) = F(B)$.

Finally, if $m \in \underline{B}^{!}$, Lemma (0.9.3) guarantees that $m = \ker(\, p(m)\colon p(T_0(m)) \longrightarrow p(z_L))$. Hence if $F^1\colon \underline{B}^{!} \longrightarrow \underline{A}$ is a kernel-preserving functor satisfying $F^1 \cdot p = F$, we have, for each $m \in \underline{B}^{!}$,

$$F^1(m) = F^1(\ker(p(m))) = \ker(F^1(p(m))) = \ker(F(m)) = F^{!}(m)\,.$$

This proves the uniqueness part of .2.; it remains only to show that $F^{!}$ does indeed preserve kernels, when they exist, provided F preserves pullbacks of the described type. Now it is easily checked that the kernel of the $\underline{B}^{!}$-morphism

$f\colon \begin{smallmatrix} A \\ m\downarrow \\ z_L \end{smallmatrix} \longrightarrow \begin{smallmatrix} C \\ n\downarrow \\ z_L \end{smallmatrix}$ is obtained by forming the pullback P of

$$A \xrightarrow{T_0(f)} C \xleftarrow{z_{LC}} z_L\,,$$

where z_{LC} is the (unique) zero map in $\underline{B}$, and taking the composite of the pullback map $P \longrightarrow A$ with $m\colon A \longrightarrow z_L$; this composite map from P to z_L is the $\underline{B}^{!}$-object which is the kernel of f, call it $\ker(f)\colon P \longrightarrow z_L$. Then:

$$F^{!}(\ker(\, f\colon m \longrightarrow n)) = F^{!}\Big(\ker(f)\Big\downarrow^{P}_{z_L}\Big) = \ker\Big(\, F(\ker(f))\Big\downarrow^{F(P)}_{F(z_L)}\Big)$$

$$= \ker(\, F(T_0(f))\colon F(A) \longrightarrow F(C)) =$$

$$= \ker\Big(\, \ker(F(m))\Big\downarrow^{F(A)}_{F(z_L)} \xrightarrow{\ker(F(T_0(f)))} \ker(F(n))\Big\downarrow^{F(C)}_{F(z_L)}\Big)$$

$$= \ker(\, F^{!}(f)\colon F^{!}(m) \longrightarrow F^{!}(n))\,.$$

The elementary diagram-chasing argument, involving the diagram

$$\begin{array}{ccccc} P & — & F^!A & — & F^!C \\ | & & | & & | \\ \mathrm{Ker} & — & FA & — & FC \\ | & & | & & | \\ 0 & — & Fz_L & — & Fz_L \end{array}$$

which is needed to establish the identifications indicated in the second line of this string of identifications will not be reproduced here.

(0.9.6) <u>Addendum</u>. Under the working hypotheses of Lemma (0.9.4), $F^!$ will preserve finite direct products if F preserves all pullbacks of pullback diagrams of the form $A \longrightarrow z_L \longleftarrow B$. For since the direct product of $m: A \longrightarrow z_L$ with $n: B \longrightarrow z_L$ is the pullback of $A \longrightarrow z_L \longleftarrow B$ equipped with the evident map to z_L (compare (0.9.2)), we see $F^!(m \times n) =$

$$= F^!\left(\underset{A \rightarrow z_L \leftarrow B}{\text{pullback of}} \longrightarrow z_L \right) =$$

$$= \ker\left(F\left(\underset{A \rightarrow z_L \leftarrow B}{\text{pullback of}} \right) \longrightarrow F(z_L) \right)$$

$$= \ker\left(\underset{FA \rightarrow Fz_L \leftarrow FB}{\text{pullback of}} \longrightarrow F(z_L) \right)$$

$$= \ker(FA \longrightarrow Fz_L) \times \ker(FB \longrightarrow Fz_L)$$

$$= F^!(m) \times F^!(n) .$$

Implications of this sort abound; we shall point no more of them out.

Remarks. There are two ways to dualise Lemma (0.9.4) -- one is to take all the functors as contravariant and use cokernels in place of kernels, the other is to assume that $\underline{\underline{B}}$ has a right zero all direct sums with which exist, and to associate to $\underline{\underline{B}}$ the category dual to $(\underline{\underline{B}}^{*})^{!}$, i.e., the category of maps from the right zero. If we apply this procedure to the category $\underline{\underline{S}}$ (recall $\underline{\underline{S}}^{!}$ is trivial), we obtain exactly the category $\underline{\underline{S}}_{*}$ of sets with base point. (0.9.4) indicates that there is, to within equivalence, but one pointed category associated to a given category with productive left zero, and that there is often but one extension $F^{!}$ for a functor F. The next lemma indicates the naturality of the passage from F to $F^{!}$; modulo set theoretic difficulties, it asserts that this passage is a functor from the category $(\!(\underline{\underline{B}}, \underline{\underline{A}})\!)$ of functors from $\underline{\underline{B}}$ to $\underline{\underline{A}}$ (the morphisms being natural transformations) to the category $(\!(\underline{\underline{B}}^{!}, \underline{\underline{A}})\!)$, provided $\underline{\underline{A}}$ has all kernels, and presents a criterion characterising this functor.

(0.9.7) Lemma. Let $\underline{\underline{A}}$ be a category with kernels (prerequisite: $\underline{\underline{A}}$ is pointed), and let $\underline{\underline{B}}$ be a category with a productive left zero z_L. If F is a functor from $\underline{\underline{B}}$ to $\underline{\underline{A}}$, notice that $F^{!}: \underline{\underline{B}}^{!} \longrightarrow \underline{\underline{A}}$ is always defined; let κ_F denote the natural transformation from $F^{!}$ to $F \cdot T_0$ determined by the canonical injections

$$\ker(F(m)) = F^{!}(m) \longrightarrow F(T_0(m)),$$

and let ξ_F be the natural transformation from $F \cdot T_0$ to $F \cdot T_1$ determined by the maps

$$F(m) \colon F(T_0(m)) \longrightarrow F(z_L) = F(T_1(m)) .$$

.1. If $\eta \colon F \longrightarrow G$ is a natural transformation between two functors from $\underline{\underline{B}}$ to $\underline{\underline{A}}$, there is a natural transformation $\eta^! \colon F^! \longrightarrow G^!$, which is uniquely determined by the requirement that for each $m \in \underline{\underline{B}}^!$ the diagram

$$\begin{array}{ccc} F^!(m) & \xrightarrow{\eta^!(m)} & G^!(m) \\ {\scriptstyle \kappa_F(m)}\downarrow & & \downarrow{\scriptstyle \kappa_G(m)} \\ F(T_0(m)) & \xrightarrow{\eta(T_0(m))} & G(T_0(m)) \end{array}$$

should commute. Moreover, if $\lambda \colon G \longrightarrow H$ is another, then $(\lambda \cdot \eta)^! = \lambda^! \cdot \eta^!$. Thus, the passage from F to $F^!$ behaves like a functor from the (perhaps illegitimate) category $(\!(\underline{\underline{B}}, \underline{\underline{A}})\!)$ to $(\!(\underline{\underline{B}}^!, \underline{\underline{A}})\!)$. Interpreting composition with T_0 and composition with T_1 similarly, ξ behaves like a natural transformation from the first to the second, and κ like its kernel.

.2. Any "functor" in the above sense from $(\!(\underline{\underline{B}}, \underline{\underline{A}})\!)$ to $(\!(B^!, A)\!)$ for which .1. holds is naturally equivalent to the passage from F to $F^!$.

<u>Proof</u>: Since $F^!(m) = \ker(F(m))$, the existence and uniqueness of each $\eta^!(m)$ are a consequence of the fact that $\ker \colon \underline{\underline{A}}^{\downarrow} \longrightarrow \underline{\underline{A}}$ is a functor (cf. (0.6.14)); that $\eta^!$ is a natural transformation follows from the uniqueness of each map $\eta^!(m)$. The interpretations are obvious, and .2. is in effect a restatement of the definition of $F^!$.

<u>Remark</u>: Lemmas (0.9.4) and (0.9.7) can be used to indicate the sense in which the adjointness statement of the introductory paragraph is to be construed. Rather than dwell overlong on this matter, however, we prefer to pass immediately to the promised application to costructures. In order that our result should be of greatest usefulness, it should incorporate a naturality statement, and to this end, the following definition, following Eilenberg's lectures in homological algebra at Columbia University, 1962-63, will be employed.

(0.9.8) <u>Definition</u>. If $\underline{\underline{A}}$ is a concrete category and $\underline{\underline{B}}$ is an arbitrary category, the <u>category</u> $Str(\underline{\underline{B}}, \underline{\underline{A}})$ <u>of</u> $\underline{\underline{A}}$-<u>structures over objects of</u> $\underline{\underline{B}}$ has for objects all triples (Q, G, ε) with $Q \in \underline{\underline{B}}$, G a contravariant functor from $\underline{\underline{B}}$ to $\underline{\underline{A}}$, and ε a natural equivalence between $\underline{\underline{B}}^Q = \underline{\underline{B}}(-, Q) : \underline{\underline{B}} \longrightarrow \underline{\underline{S}}$ and the composite

$$|G| : \underline{\underline{B}} \xrightarrow{G} \underline{\underline{A}} \xrightarrow{|\,|} \underline{\underline{S}}$$

(so that G is, in the sense of (0.5.12), an $\underline{\underline{A}}$-structure on Q). A map from one $\underline{\underline{A}}$-structure (P, F, δ) to another (Q, G, ε) is a pair (f, η) with $f \in \underline{\underline{B}}(P, Q)$ and η a natural transformation from F to G such that

$$\begin{array}{ccc} |F| & \xrightarrow{|\eta|} & |G| \\ \delta \downarrow & & \downarrow \varepsilon \\ \underline{\underline{B}}^P & \xrightarrow[\underline{\underline{B}}^f]{} & \underline{\underline{B}}^Q \end{array}$$

is a commutatitive diagram of natural transformations between functors to $\underline{\underline{S}}$.

Sending the object (Q, G, ε) of $Str(\underline{\underline{B}}, \underline{\underline{A}})$ to Q in $\underline{\underline{B}}$ and the map (f, η) to the $\underline{\underline{B}}$-morphism f defines a functor from $Str(\underline{\underline{B}}, \underline{\underline{A}})$ to $\underline{\underline{B}}$ which, by the Yoneda theory of §0.4, is readily seen to be an immersion. Notice that, while the composite functor

$$Str(\underline{\underline{B}}^*, \underline{\underline{A}}) \longrightarrow \underline{\underline{B}}^* \longrightarrow \underline{\underline{B}}$$

is contravariant, the objects of $Str(\underline{\underline{B}}^*, \underline{\underline{A}})$ can be interpreted as costructures, in the sense of (0.5.12), over objects of $\underline{\underline{B}}$, and so we are led to define the _category_ $Costr(\underline{\underline{B}}, \underline{\underline{A}})$ _of_ $\underline{\underline{A}}$-_costructures over objects of_ $\underline{\underline{B}}$ as the _dual_ of $Str(\underline{\underline{B}}^*, \underline{\underline{A}})$,

$$Costr(\underline{\underline{B}}, \underline{\underline{A}}) = (Str(\underline{\underline{B}}^*, \underline{\underline{A}}))^* ,$$

so that the underlying $\underline{\underline{B}}$-object functor $Costr(\underline{\underline{B}}, \underline{\underline{A}}) \longrightarrow \underline{\underline{B}}$ is again covariant. The _evaluation functor_ for a category of $\underline{\underline{A}}$-structures is the functor

$$ev: \underline{\underline{B}}^* \times Str(\underline{\underline{B}}, \underline{\underline{A}}) \longrightarrow \underline{\underline{A}}$$

sending the pair of objects $(B, (Q, G, \varepsilon))$ to the $\underline{\underline{A}}$-object $G(B)$; dualising both $\underline{\underline{B}}$ and $Str(\underline{\underline{B}}, \underline{\underline{A}})$, the evaluation functor yields a functor, still denoted

$$ev: (Costr(\underline{\underline{B}}, \underline{\underline{A}}))^* \times \underline{\underline{B}} \longrightarrow \underline{\underline{A}} .$$

The length of this definition is matched by its usefulness, which is a consequence of two naturality properties. The first of these is that each functor $\phi: \underline{\underline{A}} \longrightarrow \underline{\underline{A}}_1$ between concrete categories, compatible with the immersions to $\underline{\underline{S}}$ in the sense that the diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\phi} & \mathbf{A}_1 \\ & {\scriptstyle |\,|}\searrow \quad \swarrow{\scriptstyle |\,|_1} & \\ & \mathbf{S} & \end{array}$$

commutes (to within natural equivalence), gives rise to a functor from $\mathrm{Str}(\mathbf{B}, \mathbf{A})$ to $\mathrm{Str}(\mathbf{B}, \mathbf{A}_1)$ that sends (Q, G, ε) to $(Q, \phi\cdot G, \phi(\varepsilon))$ and hence makes both

$$\begin{array}{ccc} \mathbf{B}^* \times \mathrm{Str}(\mathbf{B}, \mathbf{A}) & \xrightarrow{ev} & \mathbf{A} \\ \downarrow & & \downarrow \phi \\ \mathbf{B}^* \times \mathrm{Str}(\mathbf{B}, \mathbf{A}_1) & \xrightarrow{ev} & \mathbf{A}_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Str}(\mathbf{B}, \mathbf{A}) & \searrow & \\ \downarrow & & \mathbf{B} \\ \mathrm{Str}(\mathbf{B}, \mathbf{A}_1) & \nearrow & \end{array}$$

commutative diagrams; moreover, any functor from $\mathrm{Str}(\mathbf{B}, \mathbf{A})$ to $\mathrm{Str}(\mathbf{B}, \mathbf{A}_1)$ for which that's the case must be naturally equivalent to the one just defined, which we may call $\mathrm{Str}(B, \phi)$. From this uniqueness, it follows that if $\phi_1 : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ is another functor between concrete categories, compatible with the immersions to $\mathbf{S}$, then

$$(0.9.9) \qquad \mathrm{Str}(\mathbf{B}, \phi_1)\cdot\mathrm{Str}(\mathbf{B}, \phi) \cong \mathrm{Str}(\mathbf{B}, \phi_1\cdot\phi)\,.$$

The second naturality property is that each functor $\Phi : \mathbf{B}_1 \longrightarrow \mathbf{B}$ having a right adjoint $\Psi : B \longrightarrow B_1$ gives rise to a functor $\mathrm{Str}(\Phi, \mathbf{A}) : \mathrm{Str}(\mathbf{B}, \mathbf{A}) \longrightarrow \mathrm{Str}(\mathbf{B}_1, \mathbf{A})$ making the diagrams

$$\begin{array}{ccc} \mathrm{Str}(\mathbf{B}, \mathbf{A}) & \longrightarrow & \mathbf{B} \\ {\scriptstyle \mathrm{Str}(\Phi,\, \mathbf{A})}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \mathrm{Str}(\mathbf{B}_1, \mathbf{A}) & \longrightarrow & \mathbf{B}_1 \end{array}$$

and

$$\begin{array}{ccccc} & & \mathbf{B}^* \times \mathrm{Str}(\mathbf{B}, \mathbf{A}) & & \\ & \overset{\Phi \times \mathrm{id}}{\nearrow} & & \overset{\mathrm{ev}}{\searrow} & \\ \mathbf{B}_1{}^* \times \mathrm{Str}(\mathbf{B}, \mathbf{A}) & & & & \mathbf{A} \\ & \underset{\mathrm{id} \times \mathrm{Str}(\Phi, \mathbf{A})}{\searrow} & & \underset{\mathrm{ev}}{\nearrow} & \\ & & \mathbf{B}_1{}^* \times \mathrm{Str}(\mathbf{B}_1, \mathbf{A}) & & \end{array}$$

both commute, and uniquely determined by those requirements. Namely, $\mathrm{Str}(\Phi, \mathbf{A})$ assigns to the $\mathbf{A}$-structure (Q, G, ε) the A-structure $(\Psi(Q), G\cdot\Phi, \varepsilon')$, where ε' is the composite natural equivalence

$$\mathbf{B}_1(B, \Psi(Q)) \cong \mathbf{B}(\Phi(B), Q) \overset{\varepsilon}{\cong} |G(\Phi(B))| .$$

From the uniqueness again, it follows that when $\Phi_1 : \mathbf{B}_2 \to \mathbf{B}_1$ is another functor having a right adjoint,

(0.9.10) $\quad \mathrm{Str}(\Phi_1, \mathbf{A})\cdot\mathrm{Str}(\Phi, \mathbf{A}) \cong \mathrm{Str}(\Phi\cdot\Phi_1, \mathbf{A})$.

There are corresponding results, available by duality, for costructures; notice, however, that it is a functor from $\mathbf{B}_1$ to $\mathbf{B}$ having a <u>left</u> adjoint that will induce the functor $\mathrm{Costr}(\mathbf{B}, \mathbf{A}) \longrightarrow \mathrm{Costr}(\mathbf{B}_1, \mathbf{A})$, counterpart to that above. In particular, the second naturality property fails to describe the transference of $\mathbf{A}$-costructures from $\mathbf{B}$ to $\mathbf{B}^!$, since according to Lemma (0.9.3) the only candidate for a functor from $\mathbf{B}^!$ to $\mathbf{B}$, namely T_0, has a <u>right</u> adjoint, not a left. It is for this reason that we needed Lemmas (0.9.4) and (0.9.7); with their help, we can prove the main theorem of this section. Incidentally, the central result of the next section is also a theorem whose validity the second naturality property seems at first glance to discourage.

(0.9.11) Theorem. Let $\mathbf{B}$ be a category with a productive left zero z_L, and let $\mathbf{A}$ be a pointed category with kernels, admitting a kernel-preserving immersion $|\ |_*: \mathbf{A} \longrightarrow \mathbf{S}_*$. If $(Q, G, \varepsilon) \in \mathrm{Costr}(\mathbf{B}, \mathbf{A})$, and q_o is the base point in $||G(z_L)|_*| \cong \mathbf{B}(Q, z_L)$, then there is a unique compatible natural transformation $\varepsilon^!$ such that $(q_o, G^!, \varepsilon^!) \in \mathrm{Costr}(\mathbf{B}^!, \mathbf{A})$. This defines a functor from $\mathrm{Costr}(\mathbf{B}, \mathbf{A})$ to $\mathrm{Costr}(\mathbf{B}^!, \mathbf{A})$, the only one, to within equivalence, making the diagrams

$$\begin{array}{ccc} \mathrm{Costr}(\mathbf{B}, \mathbf{A}) & \longrightarrow & \mathbf{B} \\ \downarrow & & \uparrow T_o \\ \mathrm{Costr}(\mathbf{B}^!, \mathbf{A}) & \longrightarrow & \mathbf{B}^! \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (\mathrm{Costr}(\mathbf{B}, \mathbf{A}))^* \times \mathbf{B} & \searrow^{ev} & \\ \downarrow & & \mathbf{A} \\ (\mathrm{Costr}(\mathbf{B}^!, \mathbf{A}))^* \times \mathbf{B}^! & \nearrow_{ev} & \end{array}$$

both commute; moreover, this functor, together with $\mathrm{Costr}(p, \mathbf{A}): \mathrm{Costr}(\mathbf{B}^!, \mathbf{A}) \longrightarrow \mathrm{Costr}(\mathbf{B}, \mathbf{A})$, sets up an equivalence between the categories $\mathrm{Costr}(\mathbf{B}, \mathbf{A})$ and $\mathrm{Costr}(\mathbf{B}^!, \mathbf{A})$. If $\phi: \mathbf{A} \longrightarrow \mathbf{A}_1$ is a kernel-preserving functor to a pointed category $\mathbf{A}_1$ having kernels and equipped with a kernel-preserving immersion to $\mathbf{S}_*$, and ϕ is compatible with these immersions, then the diagram

$$\begin{array}{ccc} \mathrm{Costr}(\mathbf{B}, \mathbf{A}) & \xrightarrow{\mathrm{Costr}(\mathbf{B}, \phi)} & \mathrm{Costr}(\mathbf{B}, \mathbf{A}_1) \\ \wr\!\downarrow & & \downarrow \\ \mathrm{Costr}(\mathbf{B}^!, \mathbf{A}) & \xrightarrow{\mathrm{Costr}(\mathbf{B}^!, \phi)} & \mathrm{Costr}(\mathbf{B}^!, \mathbf{A}_1) \end{array}$$

commutes.

Remark: The formulation of the second naturality property, with respect to good functors to $\mathbf{B}$, can safely be entrusted to the reader, as can the formulation of the dual theorem describing the transference of $\underline{\underline{A}}$-structures from a category with additive right zero to its associated pointed category. Observe, as an example, that since the pointed category $\underline{\underline{S}}^!$ associated to $\underline{\underline{S}}$ is the zero category, $\underline{\underline{S}}$ has no non trivial pointed costructures (i.e., $\underline{\underline{A}}$-costructures with $\underline{\underline{A}}$ pointed, having kernels, and equipped with a kernel-preserving immersion to $\underline{\underline{S}}_*$), yet the full subcategory of $\underline{\underline{S}}$ generated by non empty sets has many such, and they are not described by our theorem. Instead, they are essentially taken care of by the appropriately dualised version of the second naturality property (in the vicinity of (0.9.10)), which compares them with the $\underline{\underline{A}}$-costructures on $\underline{\underline{S}}_*$ (notice that the functor from $\underline{\underline{S}}_*$ to $\underline{\underline{S}}$ that springs at once to mind has a left adjoint).

Proof of (0.9.11): The existence and uniqueness of a compatible $\varepsilon^!$ is due to (0.9.7); that $\varepsilon^!$ is an equivalence, so that $(q_o, G^!, \varepsilon^!) \in \mathrm{Costr}(\underline{\underline{B}}^!, \underline{\underline{A}})$ follows from the identifications

$$\begin{aligned} \big||G^!(m)|_*\big| &\cong \big||\ker(\,G(m)\colon G(T(m)) \longrightarrow G(z_L)\,)|_*\big| \\ &\cong \big|\ker(\,|G(m)|_* : |G(T(m))|_* \longrightarrow |G(z_L)|_*\,)\big| \\ &\cong \{f \,/\, f \in \underline{\underline{B}}(Q, T(m)),\ m\cdot f = q_o\} \\ &\cong \underline{\underline{B}}^!(q_o, m) = (\underline{\underline{B}}^!)_{q_o}(m) . \end{aligned}$$

That the two diagrams commute is evident. Any functor from Costr($\underline{\underline{B}}$, $\underline{\underline{A}}$) to Costr($\underline{\underline{B}}'$, $\underline{\underline{A}}$) making the diagrams commute must send (Q, G, ε) to $(q_o, G', ?)$ by (0.9.4)and (0.9.7), and then ? must be ε', which proves the uniqueness. The two composibes with Costr(p, $\underline{\underline{A}}$) are equivalent to the identity, since

$$\mathrm{Costr}(B, A) \longrightarrow \mathrm{Costr}(B', A) \xrightarrow{\mathrm{Costr}(p,\ A)} \mathrm{Costr}(B, A)$$

sends (Q, G, ε) via (q_o, G', ε') to $(T_o(q_o), G' \cdot p, ??)$; but $T_o(q_o) = Q$, $G' \cdot p = G$, and ?? must be ε, by the definition of T_o and q_o, (0.9.4), and the second naturality property. That the second composite is equivalent to the identity is seen in much the same way. From this equivalence, the last assertion follows, and the proof if complete.

0.10 Equational Structures

(0.10.1) Let $\triangle = ((\alpha_\lambda)_{0 \leq \lambda < \beta})$ have rank ρ, and let Q be an object in a category $\underline{\underline{A}}$ which contains all the μ-iterated products Q^μ of Q $(0 \leq \mu \leq \rho)$. If G: $\underline{\underline{A}} \longrightarrow (\triangle)$ is a contavariant functor which is a $(\triangle)$-structure on Q, then for each $A \in \underline{\underline{A}}$ and each λ $(0 \leq \lambda < \beta)$ there is an operation $F_\lambda(A)$ on $\underline{\underline{A}}^Q(A)$,

$$F_\lambda(A): \left(\underline{\underline{A}}^Q(A)\right)^{\alpha_\lambda} \longrightarrow \underline{\underline{A}}^Q(A).$$

After the identifications

$$\left(\underline{\underline{A}}^Q(A)\right)^{\alpha_\lambda} = \left(\underline{\underline{A}}(A,\ Q)\right)^{\alpha_\lambda} = \underline{\underline{A}}(A,\ Q^{\alpha_\lambda}) = \underline{\underline{A}}^{Q^{\alpha_\lambda}}(A),$$

the fact that G is a functor indicates that, for each λ $(0 \leq \lambda < \beta)$, the operations $F_\lambda(A)$ $(A \in \underline{\underline{A}})$ are part of a natural transformation $F_\lambda: \underline{\underline{A}}^{Q^{\alpha_\lambda}} \longrightarrow \underline{\underline{A}}^Q$, which by the Yoneda correspondence (0.4.4) is determined by a unique $\underline{\underline{A}}$-morphism $f_\lambda: Q^{\alpha_\lambda} \longrightarrow Q$. By an obvious extension of the terminology of §0.7, we call these f_λ 's <u>operations of length</u> α_λ <u>on</u> Q, and we say they <u>realize</u> the given $(\triangle)$-structure G on Q, and that Q is a $\triangle$-<u>algebra in</u> $\underline{\underline{A}}$. The converse, that every $\triangle$-algebra in $\underline{\underline{A}}$ has a $(\triangle)$-structure, is obvious -- the above argument is easily reversible. Moreover, the notions $\triangle$-algebra, $\triangle$-algebra in $\underline{\underline{S}}$, and set having a $(\triangle)$-structure, all coincide.

(0.10.2) Suppose that Q is a Δ-algebra in $\underline{\underline{A}}$, and the category $\underline{\underline{A}}$ contains all iterated products Q^{μ} $(0 \leq \mu \leq \rho)$ of Q. Define a subcategory $\ddot{Q}$ of $\underline{\underline{A}}$ as follows. For each ordinal μ of cardinality $\leq \rho$, pick an iterated product Q^{μ} of Q in $\underline{\underline{A}}$ -- these are the objects of $\ddot{Q}$; the maps of $\ddot{Q}$ are generated (via composition and product-formation) by the operations f_{λ} of Q, and by the canonical projections $Q^{\mu} \longrightarrow Q$ and diagonal maps $Q \longrightarrow Q^{\mu}$. If we form, in this way, the subcategory $\ddot{W}$ of $\underline{\underline{S}}$ associated to a free Δ-algebra (in $\underline{\underline{S}}$) W generated by a set X (assumed well-ordered), then there is a 1-1 correspondence between the elements of W and the $\ddot{W}$-morphisms from W^X to W, obtained by assigning to the $\ddot{W}$-morphism $\ddot{w}: W^X \longrightarrow W$ the element $\ddot{w}(u) \in \underline{\underline{W}}$, where $u \in W^X$ is the inclusion of the generators $X \longrightarrow W$, and assigning to $w \in W$ its expression as a polynomial in the elements of X. In other words, W^X is a free $\ddot{W}$-object, with respect to the inclusion $\ddot{W} \longrightarrow \underline{\underline{S}}$, generated by a singletion -- the inclusion of the generator is the inclusion of u in W^X. All this only makes sense, of course, if $\text{card}(X) \leq \rho$.

(0.10.3) Słomiński's results in [26, Chap. III, §3] may be interpreted as stating that whenever $\text{card}(X) = \rho$, there is, for each Δ-algebra (in $\underline{\underline{S}}$) A, a functor $\Omega_A: \ddot{W} \longrightarrow \ddot{A}$ having the following properties. To the object W^{μ} is assigned the object A^{μ}; to projections and diagonal maps in $\ddot{W}$ are assigned the corresponding projections or diagonal maps in $\ddot{A}$; to the λ-th operation on W is assigned the λ-th operation on A $(0 \leq \lambda < \beta)$; and if $f \in A^X$ corresponds to $\tilde{f} \in (\triangle)(W, A)$ under the correspondence

$$A^X \cong S(X, A) \cong (\triangle)(W, A),$$

then $\Omega_A(\ddot{w})(f) = \tilde{f}(\ddot{w}(u))$ for all $\ddot{w} \in \ddot{W}(W^X, W)$, or in other words, $\Omega_A(\ddot{w})(\tilde{f}\cdot u) = \tilde{f}(\ddot{w}(u))$ for all $\tilde{f} \in (\triangle)(W, A)$. It is clear there is only one such functor Ω_A.

(0.10.4) Suppose now that Q is a $\triangle$-algebra in $\underline{\underline{A}}$, that $\underline{\underline{A}}$ contains all iterated products Q^μ of Q $(0 \leq \mu \leq \rho)$, and that W is a free $\triangle$-algebra (in $\underline{\underline{S}}$) generated by a set X of cardinality ρ. For each object $A \in \underline{\underline{A}}$, we write simply Ω_A for $\Omega_{\underline{\underline{A}}(A,Q)}$: $\ddot{W} \longrightarrow (\underline{\underline{A}}(A, Q))^{\cdot\cdot}$. For each $A \in \underline{\underline{A}}$, the covariant hom functor $\underline{\underline{A}}_A : \underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ induces (by restriction) a functor $\Omega^Q_A : \ddot{Q} \longrightarrow (\underline{\underline{A}}(A, Q))^{\cdot\cdot}$. We claim there is a functor $\Omega^Q : \ddot{W} \longrightarrow \ddot{Q}$ such that $\Omega^Q_A \cdot \Omega^Q = \Omega_A$. Indeed, if $\ddot{w} \in \ddot{W}(W^\mu, W)$, we have, for each $A \in \underline{\underline{A}}$, an element $\Omega_A(\ddot{w}) \in (\underline{\underline{A}}(A, Q))^{\cdot\cdot}(\underline{\underline{A}}(A, Q)^\mu, \underline{\underline{A}}(A, Q))$; after making the identifications

$$\underline{\underline{A}}(A, Q) = \underline{\underline{A}}^Q(A), \quad (\underline{\underline{A}}(A, Q))^\mu = \underline{\underline{A}}^{Q^\mu}(A), \quad (\underline{\underline{A}}(A, Q))^{\cdot\cdot} \subseteq \underline{\underline{S}},$$

and thinking of $\Omega_A(\ddot{w})$ as being in $\underline{\underline{S}}(\underline{\underline{A}}^{Q^\mu}(A), \underline{\underline{A}}^Q(A))$, it is clear that the collection $\Omega_A(\ddot{w})$ $(A \in \underline{\underline{A}})$ defines a natural transformation

$$\Omega(\ddot{w}) : \underline{\underline{A}}^{Q^\mu} \longrightarrow \underline{\underline{A}}^Q$$

which, by Yoneda (0.4.4), comes from a unique $\underline{\underline{A}}$-morphism $Q^\mu \longrightarrow Q$ which we shall call $\Omega^Q(\ddot{w})$. The verifications that $\Omega^Q(\ddot{w})$ is in fact a $\ddot{Q}$-morphism and that in this way a functor from $\ddot{W}$ to $\ddot{Q}$ is obtained are simple applications of the Yoneda naturality theory of §0.4 which we leave to the reader. That $\Omega^Q_A \cdot \Omega^Q = \Omega_A$ is clear from the definition.

(0.10.5) The purpose of the above discussion is to facilitate the description of $\triangle(E)$-structures in terms of properties of $(\triangle)$-structures. Under the 1-1 correspondence of (0.10.2) between the elements of the free $\triangle$-algebra W and the $\ddot{W}$-morphisms from W^X to W, each set of equations $E \subseteq W \times W$ corresponds uniquely to a set of pairs of $\ddot{W}$-morphisms, say $\ddot{E} \subseteq \ddot{W}(W^X, W) \times \ddot{W}(W^X, W)$.

(0.10.6) The condition that a $\triangle$-algebra (in $\underline{\underline{S}}$) A be a $\triangle(E)$-algebra is equivalent to the requirement that each $\triangle$-morphism from W to A identify both members of each pair in E; this is clearly equivalent to the requirement that $\Omega_A : \ddot{W} \longrightarrow \ddot{A}$ (of (0.10.3)) identify both members of each pair of $\ddot{W}$-morphisms in $\ddot{E}$. From this, we shall now show that if Q is a $\triangle$-algebra in $\underline{\underline{A}}$

and if $\underline{A}$ contains all products of Q necessary for the construction of $\ddot{Q}$, then the $\triangle$-structure realised by the operations on Q is in fact a $\triangle(E)$-structure (and we say Q is a $\triangle(E)$-algebra in $\underline{A}$) if and only if the functor $\Omega^Q : \ddot{W} \longrightarrow \ddot{Q}$ identifies both members of each pair in $\ddot{E}$. Indeed, the $\triangle$-structure realised by the operations on Q is in fact a $\triangle(E)$-structure if and only if $\Omega_A(\ddot{x}) = \Omega_A(\ddot{y})$ for each pair $(\ddot{x}, \ddot{y}) \in \ddot{E}$ and all $A \in \underline{A}$ (this is the Ω_A of (0.10.4)), which is equivalent to $\Omega(\ddot{x}) = \Omega(\ddot{y}) : \underline{A}^{Q^X} \longrightarrow \underline{A}^Q$, which is in turn equivalent to $\Omega^Q(\ddot{x}) = \Omega^Q(\ddot{y}) \in \underline{A}(Q^X, Q)$ for each pair $(\ddot{x}, \ddot{y}) \in \ddot{E}$.

(0.10.7) Let $[\ddot{W}, \underline{A}]$ be the category of product-preserving functors to $\underline{A}$ from the category $\ddot{W}$ associated in (0.10.2) to the free $\triangle$-algebra W generated by a set X of cardinality rank($\triangle$), and let $[\ddot{W}, \underline{A}]_E$ be the full subcategory consisting of those functors Ω in $[\ddot{W}, \underline{A}]$ for which $\Omega(\ddot{x}) = \Omega(\ddot{y})$ whenever $(\ddot{x}, \ddot{y}) \in \ddot{E}$, where $\ddot{E}$ is the set of pairs of $\ddot{W}$-morphisms associated in (0.10.5) to the set of equations $E \subseteq W \times W$. There is a functor $\bar{\Phi}_E : [\ddot{W}, \underline{A}]_E \longrightarrow \mathrm{Str}(\underline{A}, \triangle(E))$ sending the functor Ω to the $\triangle(E)$-structure (Q, G, ε), where $Q = \Omega(W)$, $\varepsilon = \mathrm{id}$, and G is the $\triangle$-structure on Q defined in (0.10.1) by the operations $\Omega(G_\lambda)$, where G_λ are the operations of W -- this is a $\triangle(E)$-structure by the previous paragraph (0.10.6).

(0.10.8) Lemma. The functor Φ_E sets up an equivalence between $[\ddot{W}, \underline{\underline{A}}]_E$ and the full subcategory of $Str(\underline{\underline{A}}, \Delta(E))$ consisting of those $\Delta(E)$-structures $(Q, G, \mathcal{E})$ for which all iterated products Q^{μ} $(0 \leq \mu \leq \rho)$ exist in $\underline{\underline{A}}$. Proof:

An object Q in $\underline{\underline{A}}$ is equivalent to $\Phi_E(\Omega) = \Omega(W)$ for some product-preserving functor $\Omega \in [\ddot{W}, \underline{\underline{A}}]_E$ if and only if all μ-iterated products of Q exist $(0 \leq \mu \leq \rho)$ and Q is a $\Delta(E)$-algebra in $\underline{\underline{A}}$, by the preceding.

(0.10.9) Corollary. If $\underline{\underline{A}}$ contains all the iterated products Q^{μ} $(0 \leq \mu \leq \rho)$ of all its objects Q, then Φ_E is an equivalence between $[\ddot{W}, \underline{\underline{A}}]_E$ and $Str(\underline{\underline{A}}, \Delta(E))$.

Proof: immediate.

(0.10.10) Theorem. Let $\underline{\underline{A}}$ be a category containing all μ-iterated products $(0 \leq \mu \leq \rho = \text{rank}(\Delta))$, and let $F\colon \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ be a functor which preserves all the relevant μ-iterated products, i.e., $F(Q^{\mu}) \cong (F(Q))^{\mu}$. Then F induces a unique functor $Str(F, \Delta(E))$ from $Str(\underline{\underline{A}}, \Delta(E))$ to $Str(\underline{\underline{A}}', \Delta(E))$ making the diagram

$$\begin{array}{ccc} Str(\underline{\underline{A}}, \Delta(E)) & \longrightarrow & \underline{\underline{A}} \\ {\scriptstyle Str(F,\, \Delta(E))}\downarrow & & \downarrow{\scriptstyle F} \\ Str(\underline{\underline{A}}', \Delta(E)) & \longrightarrow & \underline{\underline{A}}' \end{array}$$

commute. If $\phi\colon \Delta(E) \longrightarrow \Delta'(E')$ is any product-preserving functor compatible with the underlying sets, then the diagram

$$\begin{array}{ccc} Str(\underline{\underline{A}}, \triangle(E)) & \xrightarrow{Str(\underline{\underline{A}}, \phi)} & Str(\underline{\underline{A}}, \triangle'(E')) \\ \downarrow{\scriptstyle Str(F, \triangle(E))} & & \downarrow{\scriptstyle Str(F, \triangle'(E'))} \\ Str(\underline{\underline{A}}', \triangle(E)) & \xrightarrow[Str(A', \phi)]{} & Str(\underline{\underline{A}}', \triangle'(E')) \end{array}$$

commutes. <u>Proof</u>:

By (0.10.9), $Str(\underline{\underline{A}}, \triangle(E)) \cong [\ddot{W}, \underline{\underline{A}}]_E$. Define $Str(F, \triangle(E))$ to be the composite

$$Str(\underline{\underline{A}}, \triangle(E)) \longrightarrow [\ddot{W}, \underline{\underline{A}}]_E \xrightarrow{F\cdot} [\ddot{W}, \underline{\underline{A}}']_E \longrightarrow Str(\underline{\underline{A}}', \triangle(E)) .$$

This obviously makes the first diagram commute, and is, to within natural equivalence, the only such functor since the lower horizontal functor is an immersion. The uniqueness has as consequence the commutativity of the second diagram.

<u>Remarks</u>: i) A functor ϕ such as that occurring in (0.10.10) also gives rise to a functor $\ddot{\phi} : \ddot{W}' \longrightarrow \ddot{W}$, where $\ddot{W}'$ is the category associated to the free $\triangle'$-algebra W' generated by a set of cardinality rank($\triangle'$), and the diagram

$$\begin{array}{ccc} Str(\underline{\underline{A}}, \triangle(E)) & \xrightarrow{Str(\underline{\underline{A}}, \phi)} & Str(\underline{\underline{A}}, \triangle'(E')) \\ \uparrow & & \uparrow \\ [\ddot{W}, \underline{\underline{A}}]_E & \xrightarrow{\cdot\ddot{\phi}} & [\ddot{W}', \underline{\underline{A}}]_{E'} \end{array}$$

commutes.

ii) Of the three variations of (0.10.10) available by duality, we point out the one which will be used crucially in §1.7, and is obtained from (0.10.10) by replacing A' with its dual. It states that each contravariant functor, converting μ-iterated products to μ-iterated sums, from a category $\underline{\underline{A}}$ having all μ-iterated products to a category $\underline{\underline{A}}'$ $(0 \leq \mu \leq \text{rank}(\triangle))$ induces a unique compatible functor from $\text{Str}(\underline{\underline{A}}, \triangle(E))$ to $\text{Costr}(\underline{\underline{A}}', \triangle(E))$.

iii) The functors on structures of (0.10.10) and those of §0.9 are compatible with each other -- the uniqueness guarantees this in each case.

Chapter One

Boolean Rings and Vector Lattices

1.1 Definitions

Let $\Delta = (2, 2)$. A Δ-algebra with operations $\vee$, $\wedge$, satisfying the equations

$$x \vee x = x = x \wedge x$$

$$x \vee y = y \vee x$$

$$x \wedge y = y \wedge x$$

$$x \vee (y \vee z) = (x \vee y) \vee z$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

is called a <u>lattice</u>; a lattice satisfying in addition

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

is called <u>distributive</u>. An element 0 in a lattice satisfying $0 \wedge x = 0$ is called a <u>minimal element</u>, or a <u>zero</u>; there is at most one such, and it satisfies in addition $0 \vee x = x$.

When x, y, and z are elements of a lattice with zero, z is a <u>relative</u> <u>complement</u> <u>of</u> y <u>in</u> x if

$$z \wedge (y \wedge x) = 0 \ , \quad z \vee (y \wedge x) = x \ ;$$

in a distributive lattice with zero, relative complements are unique and are denoted (when they exist) as $z = x - y$. A <u>complementation</u> on a lattice is an additional operation $'$ of length one satisfying

$$x \wedge (y \wedge y') = z \wedge z' \ , \quad (x' \vee y')' = x \wedge y \ ;$$

then $x \wedge x'$ is a zero, call it 0, and x' is a relative complement of x in $0'$. A distributive lattice can have at most one complementation.

Lattices, distributive lattices, (distributive) lattices with zero, (distributive) lattices with complementation, (distributive) lattices with zero and a fixed (unique, in the distibutive case) choice of relative complements -- with <u>relative</u> <u>complementation</u> -- all form equational categories.

Let $\Delta' = (\omega, \omega)$, where ω is the first infinite ordinal. A Δ'-algebra with operations

$\bigvee_{i=0}^{\infty}$, $\bigwedge_{i=0}^{\infty}$, which is a (distributive) lattice when $\bigvee$ and $\bigwedge$ are defined by

$$x \vee y = \bigvee_{i=0}^{\infty} (x, y, y, \dots)$$
$$x \wedge y = \bigwedge_{i=0}^{\infty} (x, y, y, \dots),$$

and which satisfies, in addition, the equations

$$x \wedge (\bigvee_{i=0}^{\infty} (x_i \wedge x)) = \bigvee_{i=0}^{\infty} (x_i \wedge x)$$
$$x_i \wedge \bigvee_{i=0}^{\infty} (x_i) = x_i$$
$$x \vee (\bigwedge_{i=0}^{\infty} (x_i \vee x)) = \bigwedge_{i=0}^{\infty} (x_i \vee x)$$
$$x_i \vee \bigwedge_{i=0}^{\infty} (x_i) = x_i$$

is a (distributive) σ-lattice; (distributive) σ-lattices with zero, relative complementation, complementation, are defined as before, and also form equational categories.

The partial order associated to a lattice (σ-lattice) is defined by $x \leq y$ iff $x \wedge y = x$ (iff $x \vee y = y$). Each partial order on a set is associated to at most one lattice structure on that set; each lattice structure, in turn, comes from at most one σ-lattice structure. If a partial order is the partial order associated to a lattice (which comes from a σ-lattice), we say the partial order converts the set on which it is defined to a lattice (resp. σ-lattice).

A ring, every element of which is idempotent (i.e., is equal to its square), is called boolean. If a and b are any two elements of such a ring, there follow successively from the identity

$$a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$$

the relations

$$0 = ab + ba \quad \text{and} \quad 0 = a + a$$

(the second of which follows from the first by idempotence, taking $b = a$), from which we see in turn that

$$ab = ba .$$

The last two relations indicate that the notion of boolean ring coincides with the notion of commutative Z_2-algebra every element of which is idempotent.

Introduction into a boolean ring of a relation $\leq$ defined by $a \leq b$ iff $ab = a$ defines a partial order which converts the boolean ring into a distributive lattice with zero and relative complementation. Indeed,

$$a \wedge b = ab\,, \quad a \vee b = a + ab\,, \quad a - b = a + ab\,.$$

conversely, each distributive lattice with zero and relative complementation is converted into a boolean ring by the definitions

$$ab = a \wedge b\,, \quad a + b = (a \vee b) - (a \wedge b) = (a - b) \vee (b - a)\,.$$

Thus the notions of boolean ring and distributive lattice with zero and (necessarily unique) relative complemantation coincide.

In a similar way, it can be established that the notions of boolean ring with unit element, distributive lattice with complementation, and commutative Z_2-algebra with unit with each element idempotent, all coincide. Such a boolean ring is called unitary. Finally, it can be checked that any of the three interpretations for a (unitary) boolean ring yields the same definition of a morphism as another, which we call a *(unitary) boolean homomorphism*.

The element $(a-b)\vee(b-a)$ being, on the one hand, the sum $a+b$ of a with b and, on the other, the traditionally-named *symmetric difference* $a\triangle b$ of a and b, we shall henceforth denote the addition operation in a boolean ring by $\triangle$ rather than by $+$, and write $\overset{n}{\underset{i=1}{\triangle}}$ in place of $\sum_{i=1}^{n}$; however, we shall use interchangeably the notations ab and $a\wedge b$ for the product of a with b. Observe, incidentally, that $a\vee b = a\triangle b\triangle ab$ is what Jacobson [15, p. 8] would call the *circle composition* of a with b.

Let A be a subset of an arbitrary lattice B. An element a^* satisfying

$$(a \in A) \Longrightarrow (a \leq a^*)$$

$$(b \in B,\ a \leq b\ \forall\ a \in A) \Longrightarrow (a^* \leq b)$$

is called the *union*, *supremum*, or *least upper bound* of the elements of A and is denoted (if it exists) by

$$(1.1.1) \quad a^* = \overset{B}{\bigvee} A = \overset{B}{\bigvee_{a \in A}} a .$$

Similarly, an element a_* of B satisfying

$$(a \in A) \Longrightarrow (a_* \leq a)$$

$$(b \in B, \ b \leq a \ \forall \ a \in A) \Longrightarrow (b \leq a_*)$$

is called the <u>intersection</u>, <u>infimum</u>, or <u>greatest</u> <u>lower</u> <u>bound</u> of the elements of A and is denoted (if it exists) by

$$(1.1.2) \quad a_* = \overset{B}{\bigwedge} A = \overset{B}{\bigwedge_{a \in A}} a .$$

A lattice is called <u>complete</u> if all unions and intersections exist, <u>boundedly</u> <u>complete</u> if all unions (intersections) of subsets having at least one upper (lower) bound exist; a lattice homomorphism is <u>complete</u> if it preserves whatever unions and intersections exist in the domain. Incidentally, the similarity of the symbols $\overset{B}{\bigvee}$, $\overset{B}{\bigwedge}$ with the symbols $\vee$, $\wedge$ and $\overset{\infty}{\underset{i=0}{\bigvee}}$, $\overset{\infty}{\underset{i=0}{\bigwedge}}$ can lead to no error since

$$x \vee y = \overset{B}{\bigvee}\{x, y\}, \quad \overset{\infty}{\underset{i=0}{\bigvee}}(x_i) = \overset{B}{\bigvee}\{x_0, x_1, x_2, \ldots\}$$

$$x \wedge y = \overset{B}{\bigwedge}\{x, y\}, \text{ and } \overset{\infty}{\underset{i=0}{\bigwedge}}(x_i) = \overset{B}{\bigwedge}\{x_0, x_1, \ldots\} .$$

If the lattice B is clear from the context, we omit the symbol B occurring in formulae (1.1.1), (1.1.2).

An element of a boolean ring is called a <u>soma</u> (this is just as reasonable as calling an element of a Banach space a vector; of a topological space, a point; or of a differential graded module, a chain; and is moreover justified both linguistically (by Carathéodory [2, §3, p. 11]) and historically (by Götz [11, footnote one])). If a and b, then, are somas in a boolean ring, we say a is a <u>subsoma</u> of b, or b is an <u>oversoma</u> of a, if $a \leq b$; we say a and b are <u>disjoint</u> if $a \wedge b = 0$. Further, a subset A of a boolean ring consisting of pairwise disjoint somas is itself called disjoint, and the union, if it exists, of a disjoint set A of somas is called a <u>disjoint</u> <u>union</u>, and is sometimes denoted by $\bigvee\!\!\!\!\bigvee_{a \in A} a$.

The following infinite distributivity rules are valid for any subset of a distributive lattice B :

$$(1.1.3) \quad b \in B,\ \overset{B}{\bigvee_{a \in A}} a \in B \Rightarrow b \wedge \Big(\overset{B}{\bigvee_{a\in A}} a\Big) = \overset{B}{\bigvee_{a \in A}} (b \wedge a)$$

$$(1.1.4) \quad b \in B,\ \bigwedge_{a \in A}^{B} a \in B \Rightarrow b \vee \left(\bigwedge_{a \in A}^{B} a \right) = \bigwedge_{a \in A}^{B} (b \vee a)$$

When B is a boolean ring, one has also DeMorgan's rule:

$$(1.1.5) \quad \begin{cases} \text{if } a \leq b \text{ for all } a \in A, \text{ then} \\ \bigvee_{a \in A} a \in B \text{ iff } \bigwedge_{a \in A} (b - a) \in B, \text{ and} \\ b - \bigvee_{a \in A} a = \bigwedge_{a \in A} (b - a) \end{cases}$$

A (unitary) boolean ring in which all countable unions (resp. intersections) exist is called a (unitary) σ-ring (resp. (unitary) δ-ring). It follows easily from (1.1.5) that every σ-ring is a δ-ring, and hence a distributive relatively complemented σ-lattice. From DeMorgan, again, every unitary δ-ring is a σ-ring. A (unitary) boolean homomorphism between two (unitary) σ-rings (resp. δ-rings) is called a (unitary) σ-morphism (resp. δ-morphism) if it preserves countable unions (resp. countable intersections). That each σ-morphism is a δ-morphism, and hence a homomorphism of σ-lattices, and that each δ-morphism between two σ-rings is in fact a σ-morphism (more generally, that each δ-morphism preserves whatever countable unions are present in the domain) are further consequences of DeMorgan.

A boolean ring in which all intersections exist is called $\wedge$-complete; if all unions exist, complete. A boolean ring is complete if and only if it is unitary and $\wedge$-complete.

These facts are developed more extensively in Birkhoff [1], Carathéodory [2], and Sikorski [25], where, in particular, proofs of the assertions here made can be found.

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1.2 Unification

Let $\underline{\underline{A}}$ be an autonomous category with tensor products (cf. (0.5.13)), and form the subcategory $\underline{\underline{R}}(\underline{\underline{A}})$ of $\underline{\underline{A}}^{\downarrow} = \underline{\underline{\mathrm{Mor}}}(\underline{\underline{A}})$ whose objects are the $\underline{\underline{A}}$-morphisms $m: A \otimes A \longrightarrow A$ for which the diagrams (of maps in $\underline{\underline{A}}$)

$$(1.2.1)\qquad \begin{array}{ccc} A\otimes A & & \\ \big\downarrow{\scriptstyle tw} & \searrow^{m} & \\ & & A \\ & \nearrow_{m} & \\ A\otimes A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & A\otimes A & & \\ & \nearrow^{id_A\otimes m} & & \searrow^{m} & \\ A\otimes A\otimes A & & & & A\,, \\ & \searrow_{m\otimes id_A} & & \nearrow_{m} & \\ & & A\otimes A & & \end{array}$$

where tw is the twisting automorphism of $A\otimes A$, both commute, and whose morphisms $\underline{\underline{R}}(\underline{\underline{A}})\left(m \begin{array}{c} A\otimes A \\ \downarrow \\ A \end{array},\ m' \begin{array}{c} A'\otimes A' \\ \downarrow \\ A' \end{array} \right)$ are those $\underline{\underline{\mathrm{Mor}}}(\underline{\underline{A}})$-morphisms $\phi = (\phi_0, \phi_1)$ for which $\phi_0 = \phi_1 \otimes \phi_1$. Thus the target functor $T_1: \underline{\underline{A}}^{\downarrow} \longrightarrow \underline{\underline{A}}$ (cf. (0.3.9)) gives rise to an immersion $\underline{\underline{R}}(\underline{\underline{A}}) \longrightarrow \underline{\underline{A}}$ identifying $\underline{\underline{R}}(\underline{\underline{A}})\left(m \begin{array}{c} A\otimes A \\ \downarrow \\ A \end{array},\ m' \begin{array}{c} A'\otimes A' \\ \downarrow \\ A' \end{array} \right)$ with the set of those $\underline{\underline{A}}$-morphisms $f: A \longrightarrow A'$ for which the diagram

$$(1.2.2)\qquad \begin{array}{ccc} A\otimes A & \xrightarrow{f\otimes f} & A'\otimes A' \\ \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle m'} \\ A & \xrightarrow{\quad f \quad} & A' \end{array}$$

commutes. For this reason, the $\underline{\underline{R}}(\underline{\underline{A}})$-object $m: A\otimes A \longrightarrow A$

is abusively denoted A, m is called the <u>multiplication</u> on A, and we tend to think

(1.2.3) $\underline{\underline{R}}(\underline{\underline{A}})(A, A') \subseteq \underline{\underline{A}}(A, A')$.

$\underline{\underline{R}}(\underline{\underline{A}})$ is the socalled <u>category of</u> (commutative and associative) $\underline{\underline{A}}$-<u>algebras</u> (commutative and associative refer to the commutativity of diagrams (1.2.1), which of course needn't have been required). An $\underline{\underline{A}}$-algebra $m: A \otimes A \longrightarrow A$ is called <u>idempotent</u> if the composition

$$|A| \xrightarrow{d} |A| \times |A| \xrightarrow{\ddot{m}} |A|$$

is $\mathrm{id}_{|A|}$, where d is the diagonal map and $\ddot{m}$ is the underlying function of the bilinear map associated to m.

Where ${}_k\underline{\underline{M}}$ is the (concrete, autonomous, equational, pointed) category of modules over the commutative ring k, $\underline{\underline{R}}({}_k\underline{\underline{M}})$, which we denote ${}_k\underline{\underline{A}}$, is just the category of (commutative, associative) k-algebras and k-linear multiplicative homomorphisms; $\underline{\underline{R}}(\underline{\underline{AG}}) = \underline{\underline{R}}({}_Z\underline{\underline{M}}) = {}_Z\underline{\underline{A}}$ is the category of rings; $\underline{\underline{R}}(\underline{\underline{S}})$ (resp. $\underline{\underline{R}}(\underline{\underline{S}}_*)$) is the category of (comm., assoc.) monoids (resp. with zero).

An $\underline{\underline{A}}$-algebra A with multiplication m has a unit if there is a point $u_A \in |A|$ for which $\ddot{m}(a, u_A) = a$ for all $a \in |A|$; such a point is unique, if it is present, and is called the unit of A. The subcategory of $\underline{\underline{R}}(\underline{\underline{A}})$ consisting of $\underline{\underline{A}}$-algebras having units and unit-preserving $\underline{\underline{R}}(\underline{\underline{A}})$-morphisms is denoted $\hat{\underline{\underline{R}}}(\underline{\underline{A}})$; in the special case $\underline{\underline{A}} = {}_k\underline{\underline{M}}$, we write ${}_k\hat{\underline{\underline{A}}}$, and we have the usual k-algebras with unit; algebras with unit (and unit-preserving morphisms) are called unitary.

In each category ${}_k\underline{\underline{M}}$, the object k (assuming the ring k has a unit) is a free object generated by a singleton (cf. (0.5.10)), and the functor $\otimes\, k$ is naturally equivalent with $\mathrm{id}_{{}_k\underline{\underline{M}}}$. Thus ${}_k\hat{\underline{\underline{A}}}$ is the subcategory of ${}_k\underline{\underline{A}}$ whose objects $m\colon A \otimes A \longrightarrow A$ admit a (necessarily unique) ${}_k\underline{\underline{M}}$-morphism $u_A\colon k \longrightarrow A$ such that the diagram

$$\begin{array}{ccc} A\otimes k & & \\ {\scriptstyle \mathrm{id}_A \otimes u_A}\big\downarrow & \searrow^{\cong} & A \\ A\otimes A & \nearrow_{m} & \end{array}$$

commutes, and whose

morphisms ${}_k\hat{\underline{\underline{A}}}(A, A')$ are those morphisms $f \in {}_k\underline{\underline{A}}(A, A')$ which preserve units, i.e., which satisfy $f \cdot u_A = u_{A'}$. One easily checks that ${}_k\hat{\underline{\underline{A}}}(k, A) = \{u_A\}$, so that k is a left zero in ${}_k\hat{\underline{\underline{A}}}$; also, a k-algebra is idempotent iff the inclusion (1.2.3) ${}_k\underline{\underline{A}}(k, A) \subseteq {}_k\underline{\underline{M}}(k, A)$ ($\cong A$) is a 1-1 correspondence.

The category $\underline{\underline{B}}$ of boolean rings and boolean homomorphisms can now be defined as the full subcategory of $Z\underline{\underline{A}}$ or of $Z_2\underline{\underline{A}}$ whose objects are idempotent; the remaining subcategories of $Z_2\underline{\underline{A}}$ which are of interest to us are:

$\hat{\underline{\underline{B}}} = \underline{\underline{B}} \cap Z_2\hat{\underline{\underline{A}}}$: unitary boolean rings and unitary boolean homomorphisms ;

δ : δ-rings and δ-morphisms ;

σ : σ-rings and σ-morphisms ;

$\hat{\sigma} = \sigma \cap Z_2\hat{\underline{\underline{A}}}$ $(= \delta \cap Z_2\hat{\underline{\underline{A}}})$: unitary σ-rings and unitary σ-morphisms --also called $\hat{\sigma}$-rings and $\hat{\sigma}$-morphisms .

Each of the categories ${}_k\underline{\underline{M}}$, ${}_k\underline{\underline{A}}$, ${}_k\hat{\underline{\underline{A}}}$, $\underline{\underline{B}}$, $\hat{\underline{\underline{B}}}$, δ, σ, $\hat{\sigma}$ is equational; indeed, the operations and equations can be so chosen that each has countable rank and that each of the functors (indeed, immersions all) in the diagram

$$
(1.2.4)\qquad
\begin{array}{ccccccccccc}
 & & \underline{\underline{S}} & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & \underline{\underline{S}}_* & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
k^{\underline{\underline{M}}} & \longrightarrow & \underline{\underline{AG}} & \longleftarrow & Z_2^{\underline{\underline{M}}} & & & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & & & \\
k^{\underline{\underline{A}}} & \longrightarrow & Z^{\underline{\underline{A}}} & \longleftarrow & Z_2^{\underline{\underline{A}}} & \longleftarrow & \underline{\underline{B}} & \longleftarrow & \delta & \longleftarrow & \sigma \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
k^{\hat{\underline{\underline{A}}}} & \longrightarrow & Z^{\hat{\underline{\underline{A}}}} & \longleftarrow & Z_2^{\hat{\underline{\underline{A}}}} & \longleftarrow & \hat{\underline{\underline{B}}} & \longleftarrow & & \longleftarrow & \hat{\sigma}
\end{array}
\qquad (k \in Z^{\hat{\underline{\underline{A}}}})
$$

is a composition of functors of the type considered in (0.7.7), and hence <u>has a left adjoint</u>.

The notation we adopt for these functors and their left adjoints is the bare minimum: the symbol for the functor will be its range category, except that when no clarity is sacrificed, a functor in (1.2.4) will be denoted by absolute value bars or even not at all, and a left adjoint to any of the five lower vertical functors will be written as ^, and called <u>unification</u>. One should always keep in mind the earlier remark (concluding §0.6) that a left (right) functorial representation preserves left (right) representations; in particular, if each of two functors F, G has a left adjoint X, Y, then $G \cdot F$ has as left adjoint $X \cdot Y$, each of our functors $| \ |$ preserves direct products, and each of their left adjoints (in particular, unification) preserves direct sums and free objects. Should confusion be likely to arise as to which category a direct sum is being formed in, the category will accompany the symbol for the sum, as $\bigoplus_{i \in I}^{\hat{\sigma}} A_i$, for example (no such confusion can arise with products).

We remark that each of the categories in (1.2.4), with the exception of the extreme top and bottom ones, has a zero object, hence is pointed, and we may speak of kernels and cokernels of maps; these are related to quotients by equivalence relations, as hinted at in (0.5.11), as follows. If $f: A \longrightarrow B$ is a morphism, let ρ be the equivalence relation on $|A|$ determined by $|f|$, i.e., $x \rho y$ iff $f(x) = f(y)$. Then $\ker f = \{x / x \rho 0\}$. Likewise, let ρ' be the equivalence relation on $|B|$

generated by the relation ρ_0 defined by $f(x) \rho_0 0$ for all $x \in |A|$; then coker $f = B/\rho'$.

We pass to a more detailed examination of the unification functors $\hat{}: \underline{\underline{K}} \longrightarrow \hat{\underline{\underline{K}}}$ ($\underline{\underline{K}} = {}_k\underline{\underline{A}}, \underline{\underline{B}}, \sigma$). To this end, let $|A|$ mean the underlying k-module of the $\underline{\underline{K}}$-object or $\hat{\underline{\underline{K}}}$-object A (agree $k = \mathbb{Z}_2$ when $\underline{\underline{K}} = \underline{\underline{B}}, \sigma$), so that A "is" a ${}_k\underline{\underline{M}}$-morphism $m_A: |A| \otimes |A| \longrightarrow |A|$ of the type described at the beginning of this section. Define $\hat{m}_A$ to be the composite

$$(|A| \times k) \otimes (|A| \times k) \longrightarrow$$

$$\xrightarrow{\cong} (|A| \otimes |A|) \times (|A| \otimes k) \times (k \otimes |A|) \times (k \otimes k) \longrightarrow$$

$$\xrightarrow{m_A \times \cong \times \cong \times \cong} (|A| \times |A| \times |A|) \times k \xrightarrow{\mathrm{add}^n \times id_k} |A| \times k \;;$$

let $p_A \in {}_k\underline{\underline{M}}(|A| \times k,\ k)$ and $i_A \in {}_k\underline{\underline{M}}(|A|,\ |A| \times k)$ be the canonical projection and injection, respectively.

(1.2.5) <u>Unification Lemma</u>. Where $\underline{\underline{K}}$ is one of the categories ${}_k\underline{\underline{A}}, \underline{\underline{B}}, \sigma$, and $|\ |$ is the "underlying k-module" functor of (1.2.4) (take $k = \mathbb{Z}_2$ when $\underline{\underline{K}} = \underline{\underline{B}}$ or σ), define $\hat{m}_A$, p_A, i_A, as above, for all A in $\underline{\underline{K}}$. Then:

.1. $\hat{m}_A$ is in $\underline{\underline{K}}$, p_A and i_A come from $\underline{\underline{K}}$-morphisms; the canonical injection $u: k \longrightarrow |A| \times k$ is a unit, so that $\hat{m}_A$ is in $\hat{\underline{\underline{K}}}$; the canonical injection i_A, qua $\underline{\underline{K}}$-morphism, is universal for the functor $\hat{\underline{\underline{K}}} \longrightarrow \underline{\underline{K}} \xrightarrow{\underline{\underline{K}}_A} \underline{\underline{S}}$, so that $\hat{A}$ "is" $\hat{m}_A: (|A| \times k) \otimes (|A| \times k) \longrightarrow (|A| \times k)$; p_A (resp. i_A) is the universal element making k the cokernel of i_A (resp. A the kernel of p_A), so that, in particular, p_A is a $\hat{\underline{\underline{K}}}$-morphism.

.2. There is a productive left zero in $\hat{\underline{\underline{K}}}$, namely, the ground ring k, and assignation to $A \in \underline{\underline{K}}$ of p_A: $\hat{A} \longrightarrow k$ in $\hat{\underline{\underline{K}}}^!$ defines a functor

$$p: \underline{\underline{K}} \longrightarrow \hat{\underline{\underline{K}}}^!$$

which, together with the functor composite

$$\hat{\underline{\underline{K}}}^! \longrightarrow \underline{\underline{\mathrm{Mor}}}(\hat{\underline{\underline{K}}}) \longrightarrow \underline{\underline{\mathrm{Mor}}}(\underline{\underline{K}}) \xrightarrow{\ker} \underline{\underline{K}},$$

sets up an equivalence between the categories $\underline{\underline{K}}$ and $\hat{\underline{\underline{K}}}^!$.

<u>Remarks</u>: The theorems of §§0.9 and 0.10 are available to transfer costructures from $\hat{\underline{\underline{K}}}$ to $\underline{\underline{K}}$, by virtue of .2.; they will be invoked merely by reference to the present theorem. Observe also that, as a consequence of .1., the diagram

(1.2.6)
$$\begin{array}{ccccc} Z_2^{\underline{\underline{A}}} & \longleftarrow & \underline{\underline{B}} & \longleftarrow & \sigma \\ \downarrow{\hat{}} & & \downarrow{\hat{}} & & \downarrow{\hat{}} \\ Z_2^{\hat{\underline{\underline{A}}}} & \longleftarrow & \hat{\underline{\underline{B}}} & \longleftarrow & \hat{\sigma} \end{array}$$

commutes (to within natural equivalence).

<u>Proof</u>: .1. When $\underline{\underline{K}} = {}_k\underline{\underline{A}}$, this is all known, and it is easy to see that when $k = Z_2$, idempotence of the Z_2-algebra A implies idempotence of the unification $\hat{A}$. Thus the case $\underline{\underline{K}} = \underline{\underline{B}}$ is also taken care of. Finally, when $\underline{\underline{K}} = \sigma$, observe that countable unions in the (at any rate) unitary boolean ring $\hat{m}_A: (|A| \times Z_2) \otimes (|A| \times Z_2) \to (|A| \times Z_2)$ are given by the formul[illegible]

(1.2.7) $$\bigvee_{i=0}^{\infty} (a_i, \varepsilon_i) = \Big(\bigvee_{\varepsilon_i=0} a_i - \bigwedge_{\varepsilon_i=1} a_i \,,\; \bigvee_{i=0}^{\infty} \varepsilon_i \Big),$$

which also shows that p_A and i_A are σ-morphisms; this essentially finishes the proof of .1.

.2. is an immediate consequence of .1., as is the commutativity of (1.2.6).

1.3 Tensor Products

The notion of tensor product can be extended from an autonomous category $\underline{\underline{A}}$ to the category $\underline{\underline{R}}(\underline{\underline{A}})$ of (comm., assoc.) $\underline{\underline{A}}$-algebras, in such a way that the two composite functors $\underline{\underline{R}}(\underline{\underline{A}}) \times \underline{\underline{R}}(\underline{\underline{A}}) \longrightarrow \underline{\underline{A}} \times \underline{\underline{A}} \xrightarrow{\otimes} \underline{\underline{A}}$ and $\underline{\underline{R}}(\underline{\underline{A}}) \times \underline{\underline{R}}(\underline{\underline{A}}) \xrightarrow{\otimes} \underline{\underline{R}}(\underline{\underline{A}}) \longrightarrow \underline{\underline{A}}$ are naturally equivalent. For if $m_i : A_i \otimes A_i \longrightarrow A_i$ $(i = 1, 2)$ are $\underline{\underline{A}}$-algebras, define $m : (A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \longrightarrow A_1 \otimes A_2$ as the composite

$$\begin{array}{c} A_1 \otimes A_2 \otimes A_1 \otimes A_2 \\ \Big\downarrow \; id_{A_1} \otimes twist \otimes id_{A_2} \\ A_1 \otimes A_1 \otimes A_2 \otimes A_2 \\ \Big\downarrow \; m_1 \otimes m_2 \\ A_1 \otimes A_2 \end{array}.$$

Since $(m_1 \otimes m_2)(id \otimes tw \otimes id)(tw) = ((m_1 \cdot tw) \otimes (m_2 \cdot tw))(id \otimes tw \otimes id)(tw)$ $= (m_1 \otimes m_2)(tw \otimes tw)(id \otimes tw \otimes id)(tw) = (m_1 \otimes m_2)(id \otimes tw \otimes id)$, the first of diagrams (1.2.1) commutes; a similar calculation shows the other commutes too, and our assertion is validated. Moreover, if <u>both</u> A_1 and A_2 have units, so has $A_1 \otimes A_2$.

Write both $\underset{k\underline{\underline{M}}}{\otimes}$ and the associated tensor product in ${}_{k}\underline{\underline{A}}$ as $\underset{k}{\otimes}$ except when $k = Z$ or Z_2, in which cases write simply $\otimes$ and $\underset{2}{\otimes}$, respectively. The tensor product $A \underset{2}{\otimes} B$ of two boolean rings is again boolean, since the idempotence of A and B implies that of $A \underset{2}{\otimes} B$; however, no such statement is true for σ-rings or for δ-rings (unless one of them is finite). Still, examination of the situation in boolean rings will suggest an appropriate definition of tensor product for σ-rings and δ-rings. For a function $f \in \underline{\underline{S}}(A \times B, C)$, define $f_a \in \underline{\underline{S}}(B, C)$ $(a \in A)$ by $f_a(b) = f(a, b)$, and define $f^b \in \underline{\underline{S}}(A, C)$ $(b \in B)$ by $f^b(a) = f(a, b)$. If A, B, C are in ${}_{k}\underline{\underline{A}}$, then f is the bilinear map corresponding to some ${}_{k}\underline{\underline{A}}$-morphism $A \underset{k}{\otimes} B \longrightarrow C$ if and only if each f_a and each f^b not only is a ${}_{k}\underline{\underline{M}}$-morphism, but also satisfies

$$f^b(a)f^b(a') = f^{b^2}(a a')$$

$$f_a(b)f_a(b') = f_{a^2}(b b') \ .$$

Consequently, if $k = Z_2$ and A, B, C are boolean,

f corresponds to a $Z_2\underline{\underline{A}}$-morphism -- i.e., a $\underline{\underline{B}}$-morphism -- from the tensor product $A \underset{2}{\otimes} B$ to C if and only if

$$f^b(a)f^b(a') = f^{b^2}(a\,a') = f^b(a\,a'),$$

$$f_a(b)f_a(b') = f_{a^2}(b\,b') = f_a(b\,b'),$$

and each f_a, f^b is a $Z_2\underline{\underline{M}}$-morphism, i.e., iff each f_a, f^b is boolean homomorphism. Thus it is reasonable to require that the bilinear maps to be used in the category σ should be those $f \in \underline{\underline{S}}(A \times B, C)$ (A, B, and C σ-rings) for which each f_a, f^b is a σ-morphism. Before giving a formal definition, we reiterate the fact that, as a consequence of DeMorgan's rule (1.1.5), each δ-morphism between two δ-rings preserves whatever countable unions are present in the domain, so that a δ-morphism between two σ-rings is already a σ-morphism.

The rest of this section should be read twice, each occurrence of σ being replaced by δ the second time around. Let I be an index set and B_i, for each i in I, a σ-ring. Let $f \in S(B, C)$, where C is a σ-ring and $B = \bigtimes_{i \in I} B_i$, let $j \in I$, and let $b \in \bigtimes_{i \in I - \{j\}} B_i$ (by convention, I has at least two elements); define

$f_{j,b} \in \underline{\underline{S}}(B_j, C)$ by the formula

(1.3.1) $\quad f_{j,b}(b_j) = f(a)$, where $a = (a_i)_{i \in I}$ and $a_i = \begin{cases} b_i, & i \in I - \{j\} \\ b_j, & i = j \end{cases}$.

(1.3.2) Definition. An element $f \in \underline{\underline{S}}(B, C)$ is called a σ-multilinear (or σ-(card. I)-linear) map from $(B_i)_{i \in I}$ to C if

(i) each $f_{j,b}$ of (3.1.1) $\in \sigma(B_j, C)$, and

(ii) $f(\bigwedge_{s=1}^{\infty} b^s) = \bigwedge_{s=1}^{\infty} f(b^s)$, where $\left(\bigwedge_{s=1}^{\infty} b^s\right)_i = \bigwedge_{s=1}^{\infty} b_i^s$.

The set of σ-multilinear maps from $(B_i)_{i \in I}$ to C is denoted $\mathrm{Mul}((B_i)_{i \in I}, C)$ or $M_B(C)$. A left representation of M_B (which is clearly a functor $\sigma \longrightarrow \underline{\underline{S}}$) is called a σ-tensor product of the family $(B_i)_{i \in I}$ and is denoted $\bigotimes_{\sigma\, i \in I} B_i$.

Convention: $\mathrm{Mul}(B_0, C) = \sigma(B_0, C)$, $\mathrm{Mul}(\emptyset, C) = |C|$ (where $|\;|$ is underlying set) are the conventions for card $I = 1, 0$, respectively.

Remarks. Condition (ii) is a consequence of condition (i) in case the index set I is at most countable, which will be the only case of interest later. Omission of condition (ii) in the general situation leads to another notion of multilinearity, and hence to another notion of tensor product. It remains to be seen which is more useful.

The proof of the following theorem resembles the proof of the existence of direct sums in equational categories (0.7.6), and is easily adapted to the case in which condition (ii) is omitted from the definition of multilinearity.

(0.3.3) Theorem. Each family $(B_i)_{i \in I}$ has a tensor product. Proof:

The comments after (1.1.5) and the definition (1.3.2) indicate that a function $f \in \underline{\underline{S}}(B, C)$ (where again B is the direct product of all the B_i's) is σ-multilinear if and only if each of the equations below is satisfied:

1) $f(\bigwedge_{s=1}^{\infty} b^s) = \bigwedge_{s=1}^{\infty} f(b^s)$,

2) $f_{j,b}(\bigwedge_{s=1}^{\infty} b_j^s) = \bigwedge_{s=1}^{\infty} f_{j,b}(b_j^s)$,

3) $f_{j,b}(b_j^1 \triangle b_j^2) = f_{j,b}(b_j^1) \triangle f_{j,b}(b_j^2)$.

Let $\sigma : \underline{\underline{S}} \longrightarrow \mathfrak{S}$ denote the left adjoint to the underlying-set functor $\mathfrak{S} \longrightarrow \underline{\underline{S}}$ (which latter will have no name), and let $y \in \underline{\underline{S}}(B, \sigma B)$ be the universal element for σB (the inclusion of the generators. (In general, y is not $\mathfrak{S}$-multilinear.) Under the correspondence

$$M_B(C) \subseteq \underline{\underline{S}}(B, C) \cong \mathfrak{S}(\sigma B, C),$$

an element $g \in \mathfrak{S}(\sigma B, C)$ comes from an element f in $M_B(C)$ if and only if

1') $g(y(\bigwedge_{s=1}^{\infty} b^s)) = g(\bigwedge_{s=1}^{\infty} y(b^s))$,

2') $g(y_{j,b}(\bigwedge_{s=1}^{\infty} b_j^s)) = g(\bigwedge_{s=1}^{\infty} y_{j,b}(b_j^s))$,

3') $g(y_{j,b}(b_j^1 \triangle b_j^2)) = g(y_{j,b}(b_j^1) \triangle y_{j,b}(b_j^2))$;

for g corresponds to $f = g \cdot y$, and $f_{j,b} = g \cdot y_{j,b}$. Consequently, if ρ is the equivalence relation on σB generated by all the relations

1") $y(\bigwedge_{s=1}^{\infty} b^s) \rho \bigwedge_{s=1}^{\infty} y(b^s)$,

2") $y_{j,b}(\bigwedge_{s=1}^{\infty} b_j^s) \rho \bigwedge_{s=1}^{\infty} y_{j,b}(b_j^s)$,

3") $y_{j,b}(b_j^1 \triangle b_j^2) \rho (y_{j,b}(b_j^1) \triangle y_{j,b}(b_j^2))$,

and we form the quotient by ρ of σB in the category σ , we see immediately that an element $g \in \sigma(\sigma B, C)$ satisfies $g(x) = g(y)$ whenever $x \rho y$ if and only if $g \cdot y$ is σ-multilinear. Thus, the natural transformations

$$\sigma(\sigma B/\rho , C) \xrightarrow{\subseteq} \sigma(\sigma B, C)$$

$$M_B(C) \xrightarrow{\subseteq} \underline{S}(B, C) \xrightarrow{\cong} \sigma(\sigma B, C)$$

induce a 1-1 correspondence between $M_B(C)$ and $\sigma(\sigma B/\rho , C)$ which is the natural equivalence required in order that $\sigma B/\rho$ be a tensor product of the family $(B_i)_{i \in I}$.

This proves the theorem.

(1.3.4) <u>Theorem</u>. If the index set is at most countable, and if each B_i is a $\hat{\sigma}$-ring, then

$$\bigotimes_{i \in I}{}_{\sigma} B_i \cong \bigoplus_{i \in I}^{\hat{\sigma}} B_i .$$

<u>Proof</u>: Observe first of all that $\bigotimes_{\sigma\, i \in I} B_i$ has a

unit: namely, the image, under the universal element in $M_B(\bigotimes_{\sigma\, i \in I} B_i)$, of the unit element of $\bigtimes_{i \in I} B_i$. Moreover, an element $f \in M_B(C)$ gives rise to a $\hat{\sigma}$-morphism $\bigotimes_{\sigma\, i \in I} B_i \longrightarrow C$ iff condition

4) $f(1) = 1$

holds in addition to 1), 2), and 3). Thus, where $\hat{M}_B(C)$ is the subset of $M_B(C)$ consisting of those σ-multilinear maps that satisfy 4) too, we obtain an equivalence, natural in C (in $\hat{\sigma}$):

$$\hat{M}_B(C) \cong \hat{\sigma}(\bigotimes_{\sigma\, i \in I} B_i, C) .$$

By definition we have a natural equivalence

$$\hat{\sigma}(\overset{\hat{\sigma}}{\bigoplus_{i \in I}} B_i , C) \cong \bigtimes_{i \in I} \hat{\sigma}(B_i, C) ,$$

and so, to prove the theorem, it will be enought to present a natural equivalence

$$(1.3.5) \qquad \bigtimes_{i \in I} \hat{\sigma}(B_i , C) \cong \hat{M}_B(C) .$$

Accordingly, let $f \in \hat{M}_B(C)$ and let $(f_i)_{i \in I} \in \bigtimes_{i \in I} \hat{\sigma}(B_i , C)$. Define $\Phi(f) \in \bigtimes_{i \in I} \hat{\sigma}(B_i , C)$ and $\Upsilon((f_i)_{i \in I}) \in \hat{M}_B(C)$ as follows:

$$(\Phi(f))_i = f_{i,1}$$

$$\Psi((f_i)_{i \in I})((b_i)_{i \in I}) = \bigwedge_{i \in I} f_i(b_i)$$

(here the symbol "1" occurring in the expression "$f_{i,1}$" designates the unit element of $\bigtimes_{j \in I - \{i\}} B_j$). Then

$$(\Phi(\Psi((f_i)_{i \in I})))_i(b_i) = (\Psi((f_i)_{i \in I}))_{i,1}(b_i) =$$

$$= \Psi((f_i)_{i \in I})(a) \quad \left(a_j = \begin{matrix} 1, & j \neq i \\ b_i, & j = i \end{matrix}\right)$$

$$= \bigwedge_{i \in I} f_i(a_i) = f_i(b_i) ,$$

so that

$$(\Phi(\Psi((f_i)_{i \in I})))_i = f_i ,$$

whence

$$\Phi(\Psi((f_i)_{i \in I})) = (f_i)_{i \in I} ,$$

i.e.,

$\Phi \cdot \Psi$ is the identity.

Moreover,

$$\Psi(((\Phi(f))_i)_{i \in I})((b_i)_{i \in I}) = \Psi((f_{i,1})_{i \in I})((b_i)_{i \in I}) =$$

$$= \bigwedge_{i \in I} f_{i,1}(b_i) = f((b_i)_{i \in I}) ,$$

so that

$$\Psi(\Phi(f)) = \Psi(((\Phi(f))_i)_{i \in I}) = f ,$$

and so

$\Psi \cdot \Phi$ is the identity.

Thus Φ sets up a 1-1 correspondence between the sets in (1.3.5); we omit the verification of naturality, which is straightforward and completes the proof.

1. 4 Tensoring is exact

In any pointed category, a pair of maps $f: A \longrightarrow B$, $g: B \longrightarrow C$ is a short exact sequence (s.e.s.) if f is a universal element in terms of which $A = \ker g$ and g is a universal element in terms of which $C = \operatorname{coker} f$. Instead of "$(f, g)$ is a s.e.s." we also say "$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a s.e.s." In an equational pointed category, $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a s.e.s. if and only if f is 1-1, g is onto, and $g(b) = 0$ $\Longleftrightarrow$ $\exists a \in A$ such that $b = f(a)$. It need not, of course, be the case that every onto map ($\underline{\underline{S}}$-epimorphism) in an equational category is part of a s.e.s., nor that every monomorphism be. In the category of distributive lattices with minimal element, for example, there are onto maps whose kernel is the zero object, which yet are not 1-1. (E.g., let X be a set, and let $L(X)$ denote the lattice of non negative real valued functions on X; define $s: X \longrightarrow 2^X$, where 2^X is the lattice of subsets of X, by $s(f) = \{x / f(x) \neq 0\}$. Then $s(f) = \emptyset$ iff f is the minimal element of $L(X)$, yet any two functions never taking the value zero are identified under s. See Goffman [9] for further remarks.)

In the category σ, fortunately, every $\underline{\underline{S}}$-epimorphism does fit into a short exact sequence. Not every σ-monomorphism does, however. Indeed, a σ-monomorphism $f: A \longrightarrow B$ fits into a s.e.s. iff the implication

$$b \leq f(a) \Longrightarrow \exists\, a' \in A \text{ such that } b = f(a')$$

is valid for all $a \in A$ and all $b \in B$. These facts can be found in the first chapters of Carathéodory's book [2], for example. Such σ-monomorphisms are called σ-<u>ideals</u>. If $S \in \underline{\underline{S}}$ and $f \in \underline{\underline{S}}(S, B)$ $(B \in \sigma)$, the σ-ideal <u>generated by</u> f is by definition the universal element for the kernel of the universal element for the cokernel of the σ-morphism $\sigma S \longrightarrow B$ corresponding to f, or, what is the same thing, the universal element for the kernel of the canonical projection $B \longrightarrow B/\rho_f$, where

ρ_f is the equivalence relation generated by

$$f(a) \rho_f 0 , \text{ all } a \in S ,$$

and B/ρ_f is the quotient in σ of B by ρ_f .

The σ-ideal generated by f can also be described so: form the subset of B consisting of those somas b for which $\exists\, a_i \in A$ $(i = 1, 2, \ldots)$ such that $b \leq \bigvee_{i=1}^{\infty} f(a_i)$; this set is a σ-ring, the inclusion is a σ-morphism, and that is the σ-ideal generated by f . This again is extractable from Carathéodory's book, as is the fact, finally, that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a s.e.s. iff g is onto, f is an ideal, and $C \cong \operatorname{coker} f$.

The situation is similar in the category δ : every $\underline{\underline{S}}$-epimorphism in δ is a cokernel (or rather, the universal element for a kokernel), and a monomorphism in δ is (the universal element for) a kernel iff each element in the target δ-ring which is a subsoma of a countable union of images of elements of the source δ-ring (recall (0.3.9) for source and target) is itself the image of a soma in the source.

A short exact sequence $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ is said to <u>split</u> if there is a s.e.s. $0 \longrightarrow A'' \xrightarrow{f'} A \xrightarrow{g'} A' \longrightarrow 0$ such that

(1.4.1) $\qquad g \cdot f' = id_{A''}$ and $g' \cdot f = id_{A'}$.

We also say each s.e.s. splits the other. In such a case, the pair (g, g') is a universal element in terms of which $A = A' \times A''$, and, in either $\underline{\underline{B}}$, σ, or δ,

(1.4.2) $\qquad f' \cdot g(a) \triangle f \cdot g'(a) = a$ (briefly, $f' \cdot g \triangle g' \cdot f = id_A$).

Conversely, if, in any of the categories $\underline{\underline{B}}$, σ, δ, morphisms f, f', g, g' with source and target as above satisfy (1.4.1) and (1.4.2) , it follows from (0.8.4) that f and f' are monomorphisms and that g and g' are onto. Moreover, setting $a = f(a')$ in (1.4.2), for any $a' \in A'$, we have

$$f(a') = f' \cdot g(f(a')) \triangle f \cdot g'(f(a')) = f' \cdot g(f(a')) \triangle f(a') ,$$

since $g'(f(a')) = a'$, whence $f' \cdot g(f(a')) = 0$. Since f' is a monomorphism, $g(f(a')) = 0$, and so, since a' was arbitrary, $g \cdot f = 0$. A similar argument shows $g' \cdot f' = 0$. Finally, if $g(a) = 0$ $(a \in A)$, by (1.4.2) we have

$$a = f' \cdot g(a) \triangle f \cdot g'(a) = f(g'(a)),$$

which shows that $g(a) = 0 \Longrightarrow a = f(a')$ for $a' = g'(a) \in A'$. Thus $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ is a short exact sequence; similarly $0 \longrightarrow A'' \xrightarrow{f'} A \xrightarrow{g'} A' \longrightarrow 0$ is a s.e.s. and (0.4.1) shows they split each other. Arguing a little further in this spirit, one can prove (compare (0.9.5)):

(1.4.3) The following statements are equivalent.

i) S.e.ss. (f, g) and (f', g') split each other.

ii) (g, g') is the universal element (canonical projections on the factors) for a direct product.

iii) f, g, f', and g' satisfy (1.4.1) and (1.4.2).

The goal of this section is to prove the statement which is its title. To this end, we state

(1.4.4) <u>Lemma</u>. The functors $\otimes_2 B: \underline{\underline{B}} \longrightarrow \underline{\underline{B}}$ $(B \in \underline{\underline{B}})$, $\otimes_\delta B: \delta \longrightarrow \delta$ $(B \in \delta)$, and $\otimes_\sigma B: \sigma \longrightarrow \sigma$ $(B \in \sigma)$ all send split s.e.ss. to split s.e.ss. <u>Proof</u>:

Let $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ be a split s.e.s., so that there is a s.e.s. $0 \longrightarrow A'' \xrightarrow{f'} A \xrightarrow{g'} A' \longrightarrow 0$ such that (1.4.1) and (1.4.2) hold. Tensoring everything in sight with B (in the relevant category), we get maps

$$f\otimes B\colon A'\otimes B \longrightarrow A\otimes B\ , \quad f'\otimes B\colon A''\otimes B \longrightarrow A\otimes B\ ,$$

$$g\otimes B\colon A\otimes B \longrightarrow A''\otimes B\ , \quad g'\otimes B\colon A\otimes B \longrightarrow A'\otimes B\ ,$$

where $\otimes$ denotes the relevant tensor product, and we see

$$(g\otimes B)\cdot(f'\otimes B) = (g\cdot f')\otimes B = id_{A''}\otimes B = id_{A''\otimes B}\ ,$$

$$(g'\otimes B)\cdot(f\otimes B) = (g'\cdot f)\otimes B = id_{A'}\otimes B = id_{A'\otimes B}\ ,$$

$$(f'\otimes B)\cdot(g\otimes B) \triangle (f\otimes B)\cdot(g'\otimes B) = (f'\cdot g)\otimes B \triangle (f\cdot g')\otimes B$$

$$= ((f'\cdot g) \triangle (f\cdot g'))\otimes B = id_A\otimes B = id_{A\otimes B}\ .$$

Now (1.4.3) guarantees that $(f\otimes B,\ g\otimes B)$ is a s.e.s. split by the s.e.s. $(f'\otimes B,\ g'\otimes B)$, which proves the lemma.

Lemma (1.4.4) and the lemma which follows the definitions about to be made are the main tools in the proof that tensoring is exact. Let A be a boolean ring, and let a be a soma of A. Define

$$A_a = \{b\,/\,b \in A\ ,\ \ b \leq a\}\ ,$$

$$A-a = \{b\,/\,b \in A\ ,\ \ b \wedge a = 0\}\ ,$$

let f (resp. f') be the inclusion of A_a (resp. $A-a$) in A, and define $g\colon A \longrightarrow A-a$, $g'\colon A \longrightarrow A_a$ by

$$g(b) = b - a = b \triangle (a \wedge b) ,$$

$$g'(b) = a \wedge b .$$

Using (1.4.3) it is easy to prove

(1.4.5) <u>Lemma</u>. Let $\underline{\underline{K}}$ be one of the categories $\underline{\underline{B}}$, δ, σ. If a is a soma of an object A of $\underline{\underline{K}}$, then A_a and $A - a$ are objects in $\underline{\underline{K}}$, and the maps f, g, f', g' are $\underline{\underline{K}}$-morphisms; moreover, both $0 \longrightarrow A_a \xrightarrow{f} A \xrightarrow{g} A - a \longrightarrow 0$ and $0 \longrightarrow A - a \xrightarrow{f'} A \xrightarrow{g'} A_a \longrightarrow 0$ are short exact sequences in $\underline{\underline{K}}$ that split each other.

We can now state and prove the main result of this section.

(1.4.6) <u>Theorem</u>. The functors $\otimes_2 B: \underline{\underline{B}} \longrightarrow \underline{\underline{B}}$ $(B \in \underline{\underline{B}})$, $\otimes_\delta B: \delta \longrightarrow \delta$ $(B \in \delta)$, and $\otimes_\sigma B: \sigma \longrightarrow \sigma$ $(B \in \sigma)$ preserve short exact sequences and their splittings, i.e., if $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ is a short exact sequence in one of the categories $\underline{\underline{B}}$, σ, δ, and B is an object of the same category (and (f', g') splits (f, g)), then $0 \longrightarrow A' \otimes B \xrightarrow{f \otimes B} A \otimes B \xrightarrow{g \otimes B} A'' \otimes B \longrightarrow 0$ is again a s.e.s. (which is split by $(f' \otimes B, g' \otimes B)$), where $\otimes$ is the relevant tensor product.

<u>Proof</u>: This result in the case of $\bigotimes_2 B: \underline{\underline{B}} \longrightarrow \underline{\underline{B}}$ is an immediate consequence of the known corresponding result for $Z_2^{\underline{\underline{A}}}$. The proof for $\otimes_\sigma B: \sigma \longrightarrow \sigma$ will be given in detail; replacement, in this proof, of each occurrence of σ by δ, and some modifications which we shall make explicit during the proof, result in a proof for the case $\otimes_\delta B: \delta \longrightarrow \delta$.

That $(g \otimes_\sigma B) \cdot (f \otimes_\sigma B) = 0$ follows from the identity

$$(g \otimes_\sigma B) \cdot (f \otimes_\sigma B) = (g \cdot f) \otimes_\sigma B .$$

That $g \otimes_\sigma B$ is onto follows from the facts that $g \times B: A \times B \longrightarrow A'' \times B$ is onto (since g and id_B are onto) and that $\sigma(A'' \times B) \longrightarrow A'' \otimes_\sigma B$ (the canonical projection used in the proof of (1.3.4)) is onto, i.e., from the fact that $A'' \otimes_\sigma B$ is generated by the image of the universal element $A'' \times B \longrightarrow A'' \otimes_\sigma B$.

To see that $f \otimes_\sigma B$ is 1-1 and that $(g \otimes_\sigma B)(x) = 0$ $\Longrightarrow x = (f \otimes_\sigma B)(y)$ for some (necessarily unique) $y \in A' \otimes_\sigma B$, we use the following device. For each a in A', form

A'_a, and let $i_a: A'_a \longrightarrow A'$ be the inclusion. We know from (1.4.4) and (1.4.5) that <u>both maps below are monomorphisms and σ-ideals:</u>

$$(1.4.7) \quad \begin{array}{l} i_a \otimes_\sigma B: A'_a \otimes_\sigma B \longrightarrow A' \otimes_\sigma B \\ (f \cdot i_a) \otimes_\sigma B: A'_a \otimes_\sigma B \longrightarrow A' \otimes_\sigma B \longrightarrow A \otimes_\sigma B \end{array}$$

Next, where $t: A' \times B \longrightarrow A' \otimes_\sigma B$ is the universal element, let T_0 denote the set of somas in $A' \otimes_\sigma B$ of the form $a \otimes_\sigma b = t(a, b)$, and if T_λ has been defined for all ordinals $\lambda < \lambda_0$ (λ_0 a countable ordinal), define

$$T_{\lambda_0} = \{x / x \in A' \otimes_\sigma B, \ \exists \ \lambda_{mnr} < \lambda_0 \text{ and somas } x_{mnr} \text{ in } T_{\lambda_{mnr}} \text{ such that } x = \mathop{\triangle}_{m=1}^{N} \bigwedge_{n=1}^{\infty} \bigvee_{r=1}^{\infty} x_{mnr}\} .$$

(For δ-rings, the definition of T_λ ($\lambda > 0$) is to be modified by omitting the r-indexed union and the index r.) In view of the fact that T_0 generates $A' \otimes_\sigma B$, it follows from Slomiński [26, Chap. II, (2.3)] (or, on the case of σ-rings only, from Carathéodory [2, § 53]) <u>that every soma in $A' \otimes_\sigma B$ belongs to some T_λ with λ a countable ordinal</u>. From this fact, we prove by induction:

(1.4.8) Every soma in $A'\otimes_\sigma B$ is obtained as a value of some $i_a \otimes_\sigma B$. <u>Proof:</u>

Every element of T_0 is so obtained. Let λ_0 be a countable ordinal, and suppose that each soma in each T_λ ($\lambda < \lambda_0$) is so obtained. Let $x \in T_{\lambda_0}$, say $x = \underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} \bigvee_{r=1}^{\infty} x_{mnr}$ (for δ-rings, again, omit the r-indexed union), with $x_{mnr} \in T_{\lambda_{mnr}}$ and $\lambda_{mnr} < \lambda_0$; choose somas $a_{mnr} \in A'$ such that x_{mnr} is a value of $i_{a_{mnr}} \otimes_\sigma B$, say $x_{mnr} = (i_{a_{mnr}} \otimes_\delta B)(y_{mnr})$ (for δ-rings, it is harmless to assume that $x_{mn} \leq x_{m1}$ for each m, n, and one chooses somas $a_m \in A'$ ($m = 1, \ldots, N$) such that x_{m1}, and hence also x_{mn} ($n \geq 1$) is a value of $i_{a_m} \otimes_\delta B$, say $x_{mn} = (i_{a_m} \otimes_\delta B)(y_{mn})$). If a denotes the union, in A', of all the a_{mnr} (for δ-rings, of all the a_m), we see that $x_{mnr} = (i_a \otimes_\sigma B)(y_{mnr})$, so that $x = \underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} \bigvee_{r=1}^{\infty} x_{mnr}$ $= \underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} \bigvee_{r=1}^{\infty} (i_a \otimes_\sigma B)(y_{mnr}) = (i_a \otimes_\sigma B)\left(\underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} \bigvee_{r=1}^{\infty} y_{mnr}\right)$ (for δ-rings, $x_{mn} = (i_a \otimes_\delta B)(y_{mn})$, so $x = \underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} x_{mn} =$ $= \underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} (i_a \otimes_\delta B)(y_{mn}) = (i_a \otimes_\delta B)\left(\underset{m=1}{\overset{N}{\triangle}} \bigwedge_{n=1}^{\infty} y_{mn}\right)$), and we have proved the assertion (1.4.8).

With (1.4.8) we can now show $f \otimes B$ is 1-1 and an ideal. From here on, the arguments are valid for either σ-tensor product or δ-tensor product, and we write simply $\otimes$. To show $f \otimes B$ is 1-1, it suffices to show that $(f \otimes B)(b) = 0 \Longrightarrow b = 0$ $(b \in A' \otimes B)$. So assume $(f \otimes B)(b) = 0$; using (1.4.8), write $b = (i_a \otimes B)(c)$, where $a \in A'$ and $c \in A'_a \otimes B$. Since we then have

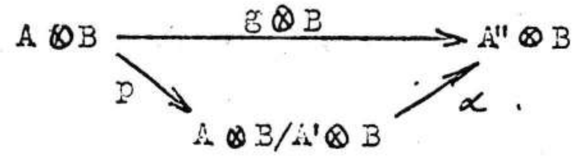
$$0 = (f \otimes B)(b) = (f \otimes B)\cdot(i_a \otimes B)(c) = ((f \cdot i_a) \otimes B)(c) ,$$

the italicised statement preceding (1.4.7) indicates that $c = 0$, so that b is indeed zero. In a similar way, we show $f \otimes B$ is an ideal: if $x \in A' \otimes B$, $y \in A \otimes B$, and $(f \otimes B)(x) \geqq y$, use (1.4.8) to write $x = (i_a \otimes B)(c)$ $(a \in A', \ c \in A' \otimes B)$; the italicised statement preceding (1.4.7) then indicates that $\exists \ d \in A'_a \otimes B$ for which

$$y = ((f \cdot i_a) \otimes B)(d) = (f \otimes B)((i_a \otimes B)(d)) ,$$

which shows $f \otimes B$ is an ideal.

We are almost done: it remains to be shown only that $g \otimes B$ is the universal element making $A'' \otimes B$ the cokernel of $f \otimes B$. The fact, mentioned at the beginning of the proof, that $(g \otimes B)(f \otimes B) = 0$, and the definition of cokernel, indicate that there is a unique map $\alpha : A \otimes B/A' \otimes B \longrightarrow A'' \otimes B$ ($A \otimes B/A' \otimes B$ denotes $\operatorname{coker}(f \otimes B)$) such that, where p is the canonical projection, the diagram



commutes. That α is onto follows from the fact, also obtained earlier, that $g \otimes B$ is. Now let β be the map corresponding to the bilinear map $\phi \in \operatorname{Mul}(A'', B; A \otimes B/A' \otimes B)$ defined by: $\phi(a'', b) = p(a \otimes b)$, where $g(a) = a''$. The defintion of ϕ is unambiguous, since if $g(a) = g(a_1) = a''$, there is a some $a' \in A'$ such that $a \triangle a_1 = f(a')$, and so $a \otimes b \triangle a_1 \otimes b = f(a') \otimes b = (f \otimes B)(a' \otimes b)$, whence $p(a \otimes b) \triangle p(a_1 \otimes b) = p(a \otimes b \triangle a_1 \otimes b) = p((f \otimes B)(a' \otimes b)) = 0$, so that $p(a \otimes b) = p(a_1 \otimes b)$. Notice $\alpha\beta(a'' \otimes b) = \alpha p(a \otimes b) = g(a) \otimes b = a'' \otimes b$, where $a'' = g(a)$. Thus:

$$\alpha\beta = \mathrm{id}_{A'' \otimes B}. \tag{1.4.9}$$

If we can show that $\beta\alpha = id_{A\otimes B/A'\otimes B}$, we shall have established the desired equivalence between $A''\otimes B$ and coker $(f\otimes B)$; since the canonical projection p from $A\otimes B$ to $A\otimes B/A'\otimes B$ = coker $(f\otimes B)$ is an epimorphism, $\beta\alpha = id_{A\otimes B/A'\otimes B}$ will follow from $\beta\alpha p = p$. Now both p and $\beta\alpha p$ correspond to bilinear maps from $A\boxtimes B$ to $A\otimes B/A'\otimes B$, say ϕ' and ϕ'', respectively, and $\beta\alpha p = p$ will follow from $\phi'' = \phi'$, which we now prove. Indeed,

$$\phi'(a, b) = p(a\otimes b), \text{ and}$$

$$\phi''(a, b) = \beta\alpha p(a\otimes b) = \beta\alpha\beta(g(a)\otimes b) =$$

$$= \beta\, id_{A''\otimes B}(g(a)\otimes b) = \beta(g(a)\otimes b) = p(a\otimes b),$$

by the definition of β and (1.4.9), and so $\phi'' = \phi'$, qed.

As a corollary, we obtain a description of $A\underset{\sigma}{\otimes}B$ which could be used as an alternate characterisation of the σ-tensor product of two σ-rings (this won't work for δ). Observe that by (1.2.5), there is, for each $\underline{\underline{K}}$-object A ($\underline{\underline{K}} = \underline{\underline{B}}$ or σ, not δ), a s.e.s.

(1.4.10) $$0 \longrightarrow A \xrightarrow[i_A]{} \hat{A} \xrightarrow[p_A]{} Z_2 \longrightarrow 0$$

where $p_A \in \hat{\underline{\underline{K}}}^{!}$ corresponds to A and i_A is the $\underline{\underline{K}}$-morphism which is the universal element for $\hat{A}$. Tensoring sequence (1.4.10) with the analogous sequence for another $\underline{\underline{K}}$-object B, we obtain the following

(1.4.11) Theorem. Let $\underline{\underline{K}} = \underline{\underline{B}}$ or $\mathcal{S}$; $\oplus$, $\otimes$, and $\times$ denote the direct sum, tensor product, direct product functors $\underline{\underline{K}} \times \underline{\underline{K}} \longrightarrow \underline{\underline{K}}$. Let $A \otimes B \longrightarrow A \oplus B$ be the $\underline{\underline{K}}$-morphism associated to the bilinear map $A \times B \longrightarrow A \oplus B$ which sends (a, b) to $j_A(a) \wedge j_B(b)$, where j_A and j_B are the injections of the summands (cf. (0.5.5)), and let $A \oplus B \longrightarrow A \times B$ correspond to the element $(id_A, 0, 0, id_B)$ in $\underline{\underline{K}}(A, A) \times \underline{\underline{K}}(A, B) \times \underline{\underline{K}}(B, A) \times \underline{\underline{K}}(B, B)$. These $\underline{\underline{K}}$-morphisms yield a short exact sequence $0 \longrightarrow A \otimes B \longrightarrow A \oplus B \longrightarrow A \times B \longrightarrow 0$.

Remark: This theorem can easily be generalised to obtain a short exact sequence of functors $0 \to \otimes \to \oplus \to \times \to 0$, in a sense which will not be made precise.

Proof: It will be convenient to abbreviate a s.e.s. $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ as $\underline{X}$; for two s.e.ss. $\underline{X}$

and $\underline{Y}$, $\underline{X} \otimes \underline{Y}$ denotes the commutative 9-gon with short exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X' \otimes Y' & \to & X' \otimes Y & \to & X' \otimes Y'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X \otimes Y' & \to & X \otimes Y & \to & X \otimes Y'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X'' \otimes Y' & \to & X'' \otimes Y & \to & X'' \otimes Y'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

and a map from $\underline{X}$ to $\underline{Y}$ is by definition three K-morphisms $X' \to Y'$, $X \to Y$, $X'' \to Y''$ making a commutative diagram

$$\begin{array}{ccccc}
X' & \to & X & \to & X'' \\
\swarrow & & \swarrow & & \swarrow \\
Y' & \to & Y & \to & Y''
\end{array}.$$

Now, given the $\underline{K}$-objects A and B, form the s.e.ss.

$$\underline{A'} : 0 \to A \xrightarrow{id} A \to 0 \to 0$$

$$\underline{B'} : 0 \to B \xrightarrow{id} B \to 0 \to 0$$

$$\underline{A} : 0 \to A \xrightarrow{i_A} \hat{A} \xrightarrow{p_A} Z_2 \to 0$$

$$\underline{B} : 0 \to B \xrightarrow{i_B} \hat{B} \xrightarrow{p_B} Z_2 \to 0$$

$$\underline{2} : 0 \to 0 \to Z_2 \xrightarrow{id} Z_2 \to 0 .$$

The triples $(0, p_A, \mathrm{id})$ and $(0, p_B, \mathrm{id})$ are maps $\underline{A} \longrightarrow \underline{2}$, $\underline{B} \longrightarrow \underline{2}$ which give rise to a map of 9-gons $\underline{A} \otimes \underline{B} \longrightarrow \underline{2} \otimes \underline{2}$, which is onto at each entry. Because of the placement of zeros in $\underline{2} \otimes \underline{2}$, it is easy to see that the kernel of this epimorphism of 9-gons, which is the 9-gon of kernels, has short exact rows and columns. Now the middle entry of $\underline{A} \otimes \underline{B}$ is $\hat{A} \otimes \hat{B}$, and by (1.3.4) and its classically known analogue for $\underline{\underline{K}} = \underline{\underline{B}}$, this middle entry is canonically isomorphis to $\hat{A} \oplus^{\hat{\underline{\underline{K}}}} \hat{B}$; hence the middle entry of the kernel 9-gon, being the kernel of the map $p_A \oplus^{\hat{\underline{\underline{K}}}} p_B : \hat{A} \oplus^{\hat{\underline{\underline{K}}}} \hat{B} \longrightarrow Z_2 \otimes Z_2 \cong Z_2 \oplus^{\hat{\underline{\underline{K}}}} Z_2 \cong Z_2$, is, by virtue of (1.2.5), just $A \oplus B$. Consequently the kernel 9-gon (having short exact rows and columns) is

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \otimes B & \longrightarrow & A \otimes \hat{B} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \hat{A} \otimes B & \longrightarrow & A \oplus B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$
,

and a standard argument for 9-gons of this type clinches the proof that the sequence we want to be a s.e.s. is.

(1.4.12) <u>Remarks</u>: .1. Using (1.4.8), we can prove that the δ-tensor product $A \otimes_\delta B$ of two σ-<u>rings</u> actually coincides with their σ-tensor product $A \otimes_\sigma B$. For each element of $A \otimes_\delta B$ being a subsoma of some element $a \otimes b$, we may assume, given a sequence $x_1, x_2, \ldots$ of somas in $A \otimes_\delta B$, that $x_i \leq a_i \otimes b_i$. But since A and B are σ-rings, each x_i is a subsoma then of $(\bigvee a_i) \otimes (\bigvee b_i)$, whence their union exists in $A \otimes_\delta B$. Thus $A \otimes_\delta B$ is a σ-ring if A and B are, and, the σ-morphisms between two σ-rings coinciding with the δ-morphisms between them, it follows easily that $A \otimes_\delta B = A \otimes_\sigma B$.

.2. An argument similar to the proof of (1.4.11), whose details we leave to the reader, establishes the following fact. If $(B_i)_{i \in I}$ is a nonvoid collection of σ-rings, then there is a short exact sequence (in which the empty sum, if it occurs, is conventionally taken as the zero ring):

$$0 \longrightarrow \bigotimes_{\sigma, i \in I} B_i \longrightarrow \bigoplus_{i \in I}^{\sigma} B_i \longrightarrow \bigtimes_{i \in I} \bigoplus_{j \in I - \{i\}}^{\sigma} B_j \longrightarrow 0 .$$

1.5 Injectives and Projectives

The $\underline{\underline{S}}$-projectives of $\mathfrak{S}$ and $\hat{\mathfrak{S}}$ are completely described by (0.8.9). In like fashion, (0.8.8) assures that the $\underline{\underline{S}}$-injectives coincide with the injectives. Nothing is known, however, about the projectives, and a description of the injectives is lacking, too. (By way of comparison, it is known (Halmos [4]) that the projectives and $\underline{\underline{S}}$-projectives in $\hat{\underline{\underline{B}}}$ coincide, or equivalently, that every epimorphism is $\underline{\underline{S}}$-epimorphism, and that a unitary boolean ring is injective in $\hat{\underline{\underline{B}}}$ if and only if it is complete; it follows easily that the same is true in the category $\underline{\underline{B}}$.) In this section we present some necessary conditions for an object of $\mathfrak{S}$ or $\hat{\mathfrak{S}}$ to be injective, and we examine two consequences of the assumption that at least one nontrivial injective exists.

Before embarking on this program, however, we present an example of an $\underline{\underline{S}}$-projective not readily afforded by (0.8.9).

(1.5.0) <u>Example</u>. Let $\mathbb{B}_{\overline{\mathbb{R}}}$ denote the σ-ring of Borel subsets of the two-point compactification $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ of the space $\mathbb{R}$ of real numbers. Let $A \xrightarrow{f} B$ be an $\underline{\underline{S}}$-epimorphism in σ, and let $\mathbb{B}_{\overline{\mathbb{R}}} \xrightarrow{k} B$ be any σ-morphism. For each extended rational number r ($r \in \{-\infty\} \cup \mathbb{Q} \cup \{+\infty\}$), put

$$F_r = k([-\infty, r])$$

and pick an element $E_r \in A$ such that $F_r = f(E_r)$. Let

$$E'_r = \bigwedge_{\substack{t \geq r, \\ t \in \mathbb{Q} \ \pm\infty}} E_t \qquad (r \in \{-\infty\} \cup \mathbb{Q} \cup \{+\infty\}) .$$

Then

$$\bigwedge_{t \geq r} E'_t = \bigwedge_{s \geq t} \bigwedge_{t \geq r} E_s = \bigwedge_{s \geq r} E_s = E'_r$$

and

$$f(E'_r) = f(\bigwedge_{t \geq r} E_t) = \bigwedge_{t \geq r} f(E_t) = \bigwedge_{t \geq r} F_t = F_r$$

(r, s, t all extended rationals), from which it follows that there is a unique σ-morphism $k': \mathbb{B}_{\overline{\mathbb{R}}} \longrightarrow A$ such that

$$E'_r = k'([-\infty, r]) ,$$

and that this k' satisfies $f \cdot k' = k$ (compare Götz [11, §2. 2.]). Another proof can be given using the Stone space techniques of §1.7 by exhibiting $\mathbb{B}_{\overline{\mathbb{R}}}$ as a retract of a Cantor space; we omit the details.

Abbreviating $(\mathbb{B}_{\overline{\mathbb{R}}})_b = \mathbb{B}_b$ for each Borel set $b \in \mathbb{B}_{\overline{\mathbb{R}}}$, the fact that $\mathbb{B}_b$ is a retract of $\mathbb{B}_{\overline{\mathbb{R}}}$ shows each $\mathbb{B}_b$ is $\underline{\underline{S}}$-projective in σ ; finally, since the unification of a σ-ring $\underline{\underline{S}}$-projective in σ is a $\hat{\sigma}$-ring $\underline{\underline{S}}$-projective in $\hat{\sigma}$ and $(\mathbb{B}_{b-\{pt\}})^{\wedge} \cong \mathbb{B}_b$, we see that each $\mathbb{B}_b$ $(b \in \mathbb{B}_{\overline{\mathbb{R}}})$ is $\underline{\underline{S}}$-projective, both in σ and in $\hat{\sigma}$.

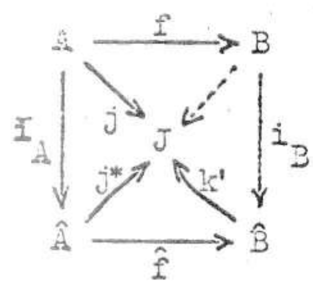
We next describe necessary conditions for a σ-ring to be injective.

(1.5.1) Lemma. In either σ or $\hat{\sigma}$ every injective is an absolute retract, and every absolute retract is complete. (In particular, every absolute retract in σ, and hence also every injective in σ, has a unit -- the union of everything in sight -- and is thus a $\hat{\sigma}$-ring!) A σ-ring is injective (resp. an absolute retract) in σ if and only if it is injective (resp. an absolute retract) in $\hat{\sigma}$. Proof:

That every injective is an absolute retract is obvious.

To prove an absolute retract is complete, we need the fact (cf. Sikorski [25, (35.1)]) that every boolean ring has associated a complete boolean ring and a monomorpphism from the given ring to the complete one preserving all possible unions. In particular, there is a monomorphism from an absolute retract to a complete ring, hence each absolute retract is a retract of a complete ring, and is therefore (cf. Halmos [14]) itself complete.

Next, let J be injective in $\hat{\sigma}$ and suppose that a monomorphism $f \in \sigma(A, B)$ and a σ-morphism $j \in \sigma(A, J)$ are given. To obtain a σ-morphism $k \in \sigma(B, J)$ such that $k \cdot f = j$, find a $\hat{\sigma}$-morphism $k' \in \hat{\sigma}(\hat{B}, J)$ such that, where j^* corresponds to j by adjointness, $k' \cdot \hat{f} = j^*$. Take $k = k' \cdot i_B$ (where



i_B is as in (1.2.5)); then $k \cdot f = k' \cdot i_B \cdot f = k' \cdot \hat{f} \cdot i_A = j^* \cdot i_A = j$, as required for J to be injective in σ. The converse

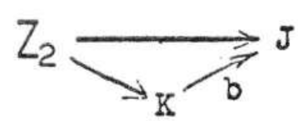
The converse is trivial because given $\hat{\sigma}$-morphisms j and f as above with J injective in σ, the σ-morphism k satisfying $j = k \cdot f$ is already a $\hat{\sigma}$-morphism. Indeed, $k(1) = k(f(1)) = j(1) = 1$, which is all that need be shown.

Finally, let J be an absolute retract in $\hat{\sigma}$, and let $f \in \sigma(J, A)$ be a monomorphism. Let $a = f(1)$, $p_a: A \longrightarrow Aa$ the canonical projection. Then $p_a \cdot f \in \hat{\sigma}(J, A_a)$ and $p_a \cdot f$ is a monomorphism. Hence there is a $\hat{\sigma}$-morphism $h: A_a \longrightarrow J$ such that $h \cdot p_a \cdot f = id_J$. Letting $g = h \cdot p_A$, we then have $g \cdot f = h \cdot p_a \cdot f = id_J$, which shows J is an absolute retract in σ. The converse is again a triviality, and the lemma is proved.

Remark: In view of this lemma, it matters little if we work in the category σ or $\hat{\sigma}$, examining our injectives.

The next lemma depends upon the fact that there is at least one $\hat{\sigma}$-ring K with no $\hat{\sigma}$-morphism to Z_2. Such rings are afforded in profusion by the theory of measurable cardinals (cf. [7, Chap. 12]). Precisely, let X be an uncountable set whose cardinality precedes the first cardinal inaccessible from $\aleph_0$ (for definitions and properties of such cardinals, cf. Ulam [31, §§1,8,9], Sikorski [25, §26], and [7, Chap. 12]), and let $K = 2^X/\rho$, where ρ is the equivalence relation "differs by at most countably many points from;" there are infinitely many such K, and none of them has a $\hat{\sigma}$-morphism to Z_2. Remark, before passing to the next lemma, that for a $\hat{\sigma}$-ring the notions $\hat{\sigma}$-morphism to Z_2 and non zero σ-morphism to Z_2 are equivalent.

(1.5.2) Lemma. There is no $\hat{\sigma}$-morphism from an injective to Z_2 (equivalently, no non zero σ-morphism).

Proof: If J is injective and $a \in \hat{\sigma}(J, Z_2)$, let $Z_2 \longrightarrow J$ and $Z_2 \longrightarrow K$ be the unique $\hat{\sigma}$-morphisms, and let $b: K \longrightarrow J$ be a $\hat{\sigma}$-morphism, available since J is injective, making the diagram  commute. Then $a \cdot b$ is a $\hat{\sigma}$-morphism from K to Z_2, which is impossible.

(1.5.3) Corollary (Halmos [14, Prob. 3]). No injective σ-ring has an atom. Proof:

To have an atom is to be a direct product $A \times Z_2$, which implies to have a canonical projection to Z_2, which is incompatible with the assumption of injectivity.

(1.5.4) Corollary. There is no non zero σ-morphism from an injective to the σ-ring 2^X. Proof:

If $a: J \longrightarrow 2^X$ is a σ-morphism, then whenever $\phi \in \sigma(2^X, Z_2)$, the composite $\phi \cdot a$ is zero, by (1.5.2). For each point x in X, define $\phi_x \in \sigma(2^X, Z_2)$ by

$$\phi_x(S) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

and observe that $\phi_x(S) = 0$ for all $x \in X$ iff $S = \emptyset$. From $\phi \cdot a = 0$ for all $\phi \in \sigma(2^X, Z_2)$ we obtain $\phi_x(a(j)) = 0$ for all $x \in X$ and all $j \in J$, whence $a(j) = \emptyset$ for all $j \in J$, which shows $a = 0$.

(1.5.5) Corollary. If J is a non zero injective, J is uncountable. Proof:

If J is finite, it admits a monomorphism to 2^X with $X = \underset{\sim}{S}(J, Z_2)$; for this map to be zero, J must be zero, which contradicts the hypothesis. So J is infinite;

but an infinite $\hat{\sigma}$-ring is uncountable [25, §20 E)].

(1.5.6) Corollary. Every σ-morphism from an injective to an $\underline{\underline{S}}$-projective is zero. Proof:

Sikorski's construction [25, (31.2)] shows that each free $\hat{\sigma}$-ring is a $\hat{\sigma}$-field of sets; each free σ-ring is therefore also a σ-field of sets, and an $\underline{\underline{S}}$-projective, being a retract of a free, therefore has a monomorphism to 2^X for some X. Composing this with any σ-morphism from an injective gives zero, by (1.5.4), which ends the proof.

(1.5.7) Corollary. No non zero injective can be an $\underline{\underline{S}}$-projective, either in σ or in $\hat{\sigma}$, let alone be an absolute retract or a projective. Proof: Immediate.

Now assume there is a non trivial injective either in σ or in $\hat{\sigma}$; an application of (1.5.1), if necessary, assures there is then a non trivial injective, say J, in $\hat{\sigma}$. It is easily seen that the four element boolean ring $G = \{0, \text{pt.}, 1 \triangle \text{pt.}, 1\}$ is a free $\hat{\sigma}$-ring generated by a single point. Let a soma a in a $\hat{\sigma}$-ring A be called non trivial if $0 \neq a \neq 1$ (so only 0, 1 are trivial); then a is non trivial if and only if the $\hat{\sigma}$-morphism $\ddot{a}: G \longrightarrow A$ corresponding by adjointness to the inclusion $\{a\} \subseteq A$ is a monomorphism. Let $h: G \longrightarrow J$ be any $\hat{\sigma}$-morphism, and let $g_a: A \longrightarrow J$ be a $\hat{\sigma}$-morphism (available since J is injective) making the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ddot{a}} & A \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g_a} & \\ & J & \end{array}$$

commute (a non trivial soma in A). The family $(g_a)_{a \text{ non trivial in } A}$ defines a $\hat{\sigma}$-morphism

$$g: A \longrightarrow \bigtimes_{\substack{a \in A \\ 0 \neq a \neq 1}} J_a \overset{\text{def}}{=} J^A \quad (\text{each } J_a \text{ is } J)$$

which we shall show, under the assumption that $0 \neq A \neq Z_2$, is a monomorphism. Indeed, the identity $g_a(a) = a$ assures that any soma a for which $g(a) = 0$ must be trivial, and $g(1) = 1$ shows that this trivial soma a cannot be 1; so $a = 0$, and g is a monomorphism. Since J^A is easily seen to be injective when $0 \neq A \neq Z_2$, we see that each such A admits a monomorphism (in $\hat{\sigma}$) to an injective. The same is trivially true for 0 and Z_2.

Finally, if B is a σ-ring, the composite of the universal element $i_B: B \longrightarrow \hat{B}$ with a monomorphism (in $\hat{\sigma}$) from $\hat{B}$ to an injective yields a monomorphism from B to an injective; so, saying a category <u>has enough injectives</u> if each object admits a monomorphism to an injective, we have thus proved:

(1.5.8) <u>Lemma</u>. If either σ or $\hat{\sigma}$ has one non trivial injective, then both σ and $\hat{\sigma}$ have enough injectives.

(1.5.9) <u>Corollary</u>. If there is a non trivial injective, then the functors $\overset{\sigma}{\oplus} B: \sigma \longrightarrow \sigma$ and $\underset{\sigma}{\otimes} B: \sigma \longrightarrow \sigma$ preserve monomorphisms. <u>Proof</u>:

Suppose $f: A' \longrightarrow A$ is a monomorphism. Let $z': A' \overset{\sigma}{\oplus} B \longrightarrow J$ be a monomorphism to an injective (possible by (1.5.8)) corresponding to a pair of maps $x': A' \longrightarrow J$, $y: B \longrightarrow J$. Find a σ-morphism $x: A \longrightarrow J$ such that $x \cdot f = x'$, and let $z: A \overset{\sigma}{\oplus} B \longrightarrow J$ be the map corresponding to the pair x, y; then consider the diagram overleaf, where the maps * and ** are defined in (1.4.11).

$$\begin{array}{ccccc} A'\otimes_{\sigma} B & \xrightarrow{*} & A'\ominus^{\sigma} B & & \\ {\scriptstyle f\otimes_{\sigma} B}\downarrow & & {\scriptstyle f\ominus^{\sigma} B}\downarrow & \searrow^{z'} & \\ & & & & J \\ & & & \nearrow_{z} & \\ A\otimes_{\sigma} B & \xrightarrow{**} & A\ominus^{\sigma} B & & \end{array}$$

The triangle commutes by the definition of z. The square commutes by the comment preceding the proof of (1.4.11). Since z' is a monomorphism, so is $f\ominus^{\sigma}B$; since z' and $*$ are monomorphisms, so is $f\otimes_{\sigma}B$. (How much easier this was than the relevant portion of the proof of (1.4.6)!)

The last result of this section asserts

(1.5.10) Theorem. If there is a non trivial injective, then the projectives and the $\underline{\underline{S}}$-projectives coincide. Proof:

Let us work in the category $\hat{\sigma}$. Assume given a $\hat{\sigma}$-morphism $f: A \longrightarrow B$ which is not an $\underline{\underline{S}}$-epimorphism, so that there is an element $b_o \in B$ not covered by f (i.e., $f(a) \neq b_o$ for all somas a in A). Let

$$X = \{a / a \in A,\ f(a) \leq b_o \text{ or } f(a) \leq 1 \triangle b_o \overset{\text{def}}{=} b'_o\},$$

$$Y = \left\{b / b \in B,\ b \leq \bigvee_{i=1}^{\infty} f(a_i),\ a_i \in X\right\},$$

$$Z = \{a / a \in A,\ a \le \bigvee_{i=1}^{\infty} a_i\ ,\ a_i \in X\}.$$

Y *and* Z *are* *σ-ideals* in the rings B and A, respectively; *if* $a \in Z$, *then* $f(a) \in Y$ (indeed, $a \in Z \Longrightarrow a \le \bigvee_{i=1}^{\infty} a_i\ (a_i \in X) \Longrightarrow f(a) \le \bigvee_{i=1}^{\infty} f(a_i)\ (a_i \in X)$); and *if* $f(a) \in Y$, *then* $a \in Z$ (indeed, $f(a) \in Y \Longrightarrow$ $f(a) \le \bigvee_{i=1}^{\infty} f(a_i)\ (a_i \in X) \Longrightarrow f(a) = \bigvee_{i=1}^{\infty} f(a \wedge a_i) \Longrightarrow$ $f(a \triangle \bigvee_{i=1}^{\infty} (a \wedge a_i)) = 0 \le b_0 \Longrightarrow (a \triangle \bigvee_{i=1}^{\infty} (a \wedge a_i)) \in X$; but where $a_0^* = a \triangle \bigvee_{i=1}^{\infty} (a \wedge a_i)$ and $a_i^* = a \wedge a_i\ (i \ge 1)$, we have $a_i^* \in X\ (i \ge 0)$ and $a = a_0^* \triangle \bigvee_{i=1}^{\infty} a_i^* \le \bigvee_{i=0}^{\infty} a_i^*$, so $a \in Z$). Consequently, if $p: A \longrightarrow A/Z$ and $q: B \longrightarrow B/Y$ are the canonical projections, there is a unique $\hat{\sigma}$-morphism $f_1: A_1 \overset{\text{def}}{=} A/Z \longrightarrow B/Y \overset{\text{def}}{=} B_1$ making the diagram $\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow p & & \downarrow q \\ A_1 & \longrightarrow & B_1 \end{array}$ commute, and f_1 is a *monomorphism*.

Next, let $b_1 = q(b_0) \in B_1$, and let $a_1 \in A_1$ satisfy $0 \ne a_1 \ne 1$. We claim that $f_1(a_1) \le b_1$ is impossible, as is $f_1(a_1) \le 1 \triangle b_1$. Indeed, if $f_1(a_1) \le b_1$, choose $a \in A$ such that $p(a) = a_1$; then there is $b \in Y$, say $b \le \bigvee_{i=2}^{\infty} f(a_i)$,

$a_i \in X$ $(i \geq 2)$, such that $f(a) \leq b_0 \vee b$; we may further assume that $f(a_i) \leq b'_0 = 1 \triangle b_0$ $(i \geq 2)$, since any a_i with $f(a_i) \leq b_0$ is unnecessary in b. Forming $a' =$ $= a - \bigvee_{i=2}^{\infty} a_i$, we have

$$f(a') = f(a - \bigvee_{i=2}^{\infty} a_i) = f(a \triangle \bigvee_{i=2}^{\infty} (a \wedge a_i) =$$
$$= f(a) \triangle \bigvee_{i=2}^{\infty} f(a) \wedge f(a_i) = f(a) - \bigvee_{i=2}^{\infty} f(a_i) \leq$$
$$\leq (b_0 \vee b) - \bigvee_{i=2}^{\infty} f(a_i) \leq b_0 .$$

Thus $a' \in Z$, and $p(a') = 0$. On the other hand,

$a_i \in X$ $(i \geq 2) \Longrightarrow p(a_i) = 0$ $(i \geq 2) \Longrightarrow p(a') = p(a)$,

for both are equal to $p(a) - \bigvee_{i=2}^{\infty} p(a_i)$. Consequently,

$a_1 = p(a) = p(a') = 0$, which contradicts the hypothesis. An argument contradicting $f_1(a_1) \leq 1 \triangle b_1$ is similar.

Thus we have a monomorphism $f_1 \in \hat{\sigma}(A_1, B_1)$ and an element b_1 of B_1 for which

$$0 \neq a_1 \in A_1 \Longrightarrow \begin{cases} f_1(a_1) \wedge b_1 \neq 0 \\ f_1(a_1) \wedge (1 \triangle b_1) \neq 0 \end{cases}$$

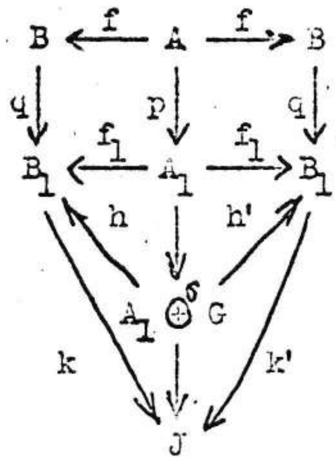
-- we say b_1 is <u>independent</u> from A_1.

Finally, let G be the four element $\hat{\sigma}$-ring that is the free $\hat{\sigma}$-ring on a single generator, and let

$g\colon G \longrightarrow B_1$, $g'\colon G \longrightarrow B_1$ be the $\hat{\sigma}$-morphisms that correspond to the inclusions $\{b_1\} \subseteq B_1$, $\{1 \triangle b_1\} \subseteq B_1$. According to Sikorski [25, (38.2)], the independence of b_1 from A_1 guarantees that the $\hat{\sigma}$-morphisms $h\colon G \overset{\hat{\sigma}}{\oplus} A_1 \longrightarrow B_1$ and $h'\colon G \overset{\hat{\sigma}}{\oplus} A_1 \longrightarrow B_1$ corresponding to the pairs of maps (g, f_1) and (g', f_1), respectively, are monomorphisms. Find a monomorphism from $G \overset{\hat{\sigma}}{\oplus} A_1$ to an injective J (possible by (1.5.8)), and find maps k and k' from B_1 to J making both

$$\begin{array}{ccc} G + A_1 & \xrightarrow{h} & B_1 \\ \downarrow & \swarrow_{k} & \\ J & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G + A_1 & \xrightarrow{h'} & B_1 \\ \downarrow & \swarrow_{k'} & \\ J & & \end{array}$$

commutative diagrams. Since $k(b_1) = k'(1 \triangle b_1) = 1 \triangle k'(b_1)$, $k \neq k'$ unless $B = \{0\}$, which is trivially impossible; then since q is an epimorphism, $k \cdot q \neq k' \cdot q$; yet commutativity of the diagram overleaf clearly indicates that $k \cdot q \cdot f = k' \cdot q \cdot f$. Thus f is not an epimorphism, in other words, every epimorphism in $\hat{\sigma}$ is onto. An application of (0.8.8) completes the proof, in the category $\hat{\sigma}$.



Pass to the category σ . Let $f \in \sigma(A, B)$ be an epimorphism. Then the unification $\hat{f} \in \hat{\sigma}(\hat{A}, \hat{B})$ is an epimorphism in $\hat{\sigma}$ (indeed, $g\cdot\hat{f} = h\cdot\hat{f}$ ($g, h \in \hat{\sigma}(\hat{B}, C)$) $\Longrightarrow g\cdot f = h\cdot f \Longrightarrow g = h$), hence onto, hence f is onto, andanother application of (0.8.B) completes the proof of the theorem, and closes the section.

1.6 Step functions

This section is devoted to the construction of the group of step functions over a boolean ring, with values, or coefficients, in an abelian group. The procedure will be to obtain a step function functor $\#: \underline{\underline{AG}} \times \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$ as the left functorial representation (recall (0.5.1)) for a certain functor $\text{Mix}: (\underline{\underline{AG}} \times \underline{\underline{B}})^* \times \underline{\underline{AG}} \longrightarrow \underline{\underline{S}}$. A remark of Freyd [5] will help show that there is a unique "extension" of the step function functor to a functor ${}_k\underline{\underline{M}} \times \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{M}}$ (where k is a commutative ring with unit and ${}_k\underline{\underline{M}}$ is the category of k-modules), such that the diagram of functors

$$(1.6.0) \qquad \begin{array}{ccc} {}_k\underline{\underline{M}} \times \underline{\underline{B}} & \longrightarrow & {}_k\underline{\underline{M}} \\ \downarrow | \; | \times \text{id}_{\underline{\underline{B}}} & & \downarrow | \; | \\ \underline{\underline{AG}} \times \underline{\underline{B}} & \underset{\#}{\longrightarrow} & \underline{\underline{AG}} \end{array}$$

(where $| \; |$ denotes the underlying group functor) commutes (at least to within natural equivalence); this in turn has an extension to a functor ${}_k\underline{\underline{A}} \times \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{A}}$. We begin with the description of the functor Mix, and an associated functor $\text{mix}: \underline{\underline{B}}^* \times \underline{\underline{AG}} \longrightarrow \underline{\underline{S}}$.

(1.6.1) Let $M, M' \in \underline{\underline{AG}}$, $B \in \underline{\underline{B}}$. A function

$$f_*: B \longrightarrow M' \qquad (\text{i.e.,} \;\; f_* \in \underline{\underline{S}}(B, M'))$$

is called _mixed_ if

$$f_*(a \vee b) = f_*(a) + f_*(b) - f_*(a \wedge b) .$$

Define $\text{mix}(B, M')$ to be the subset of $\underline{\underline{S}}(B, M')$ that consists entirely of mixed functions. Next, for each function $f: M \times B \longrightarrow M'$ (i.e., $f \in \underline{\underline{S}}(M \times B, M')$), define $f^b \in \underline{\underline{S}}(M, M')$ $(b \in B)$ and $f_m \in \underline{\underline{S}}(B, M')$ $(m \in M)$ by the formulae

$$f^b(m) = f(m, b) \qquad (m \in M) ,$$
$$f_m(b) = f(m, b) \qquad (b \in B) .$$

Finally, $\mathrm{Mix}(M, B; M')$ is by definition the subset of $\underline{\underline{S}}(M \times B, M')$ consisting of all those functions f (called once again <u>mixed</u>) for which both

$$f^b \in \underline{\underline{AG}}(M, M') \quad \text{and} \quad f_m \in \mathrm{mix}(B, M')$$

for all $b \in B$ and all $m \in M$.

It is no trouble to see that if

$$x \in \underline{\underline{AG}}(M_o, M) , \qquad y \in \underline{\underline{B}}(B_o, B) ,$$
$$z \in \underline{\underline{AG}}(M', M'_o) , \qquad f_* \in \mathrm{mix}(B, M') ,$$
$$f \in \mathrm{Mix}(M, B; M') ,$$

then $z \cdot f_* \cdot y \in \mathrm{mix}(B_o, M'_o)$, $(z \cdot f \cdot (x, y))_{m_o} = z \cdot f_{x(m_o)} \cdot y$, for $m_o \in M_o$, so that $(z \cdot f \cdot (x, y))_{m_o} \in \mathrm{mix}(B_o, M'_o)$, and $(z \cdot f \cdot (x, y))^{b_o} = z \cdot f^{y(b_o)} \cdot y$, so that also $(z \cdot f \cdot (x, y))^{b_o} \in \underline{\underline{AG}}(M_o, M'_o)$, for $b_o \in B_o$. Consequently, we have proved and may state

(1.6.2) <u>Lemma</u>. Both mix and Mix are functors, from $\underline{\underline{B}}^* \times \underline{\underline{AG}}$ and $(\underline{\underline{AG}} \times \underline{\underline{B}})^* \times \underline{\underline{AG}}$, respectively, to $\underline{\underline{S}}$.

We now reduce the problem of finding a left functorial representation for Mix to that of finding one for mix. Indeed, if mix: $\underline{\underline{B}}^* \times \underline{\underline{AG}} \longrightarrow \underline{\underline{S}}$ has a left functorial representation $F: \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$, so that there is a natural equivalence

$$\mathrm{mix}(B, M') \cong \underline{\underline{AG}}(F(B), M'),$$

the fact that $\underline{\underline{AG}}$ is autonomous indicates that this equivalence can be lifted to an equivalence of functors to $\underline{\underline{AG}}$ (and in particular, that mix may be viewed as a functor to $\underline{\underline{AG}}$). Since $\underline{\underline{AG}}$ has tensor products (cf. (0.5.13) and §1.3; tensor products in $\underline{\underline{AG}}$ are $\otimes$), if we think of mix as lifted to $\underline{\underline{AG}}$, we have

$$\begin{aligned}\mathrm{Mix}(M, B; M') &\cong \underline{\underline{AG}}(M, \mathrm{mix}(B, M')) \cap \mathrm{mix}(B, \underline{\underline{AG}}(M, M')) \\ &\cong \underline{\underline{AG}}(M, \underline{\underline{AG}}(F(B), M')) \cap \underline{\underline{AG}}(F(B), \underline{\underline{AG}}(M, M')) \\ &\cong AG(M \otimes F(B), M'),\end{aligned}$$

which proves

(1.6.3) Lemma. If the functor mix has a left functorial representation F, then the functor Mix has a left functorial representation $\#: \underline{\underline{AG}} \times \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$ given by

$$M \# B = M \otimes F(B).$$

Thus, in order to prove the existence of a left functorial representation for Mix, it suffices to do that for mix, and for this, by (0.6.1), it is enough to prove that each functor mix(B, _) has a left representation.

(1.6.4) Lemma. Let B be a boolean ring and let G: $\underline{\underline{AG}} \longrightarrow \underline{\underline{S}}$ be the functor mix(B, –). This functor G has a left representation.

Proof: Form the free abelian group AG(B) generated by the underlying set of B, so that

$$\underline{\underline{S}}(B, M') \cong \underline{\underline{AG}}(AG(B), M').$$

Let ρ be the equivalence relation on AG(B) generated by the relations

$$(a \vee b) + (a \wedge b) \ \rho \ (a) + (b) \qquad (a, b \in B)$$

where + is the addition in AG(B) and (x) is the element in AG(B) corresponding to the element $x \in B$. Define $FB = AG(B)/\rho$, the quotient in $\underline{\underline{AG}}$ of AG(B) by ρ. Observe that when $f \in \underline{\underline{S}}(B, M')$ corresponds to $f' \in \underline{\underline{AG}}(AG(B), M')$, f is mixed if and only if $x \rho y \Longrightarrow f'(x) = f'(y)$. Consequently

$$G(M') = mix(B, M') \cong \underline{\underline{AG}}(AG(B)/\rho, M'),$$

$AG(B)/\rho$ represents G, and the lemma is proved.

As remarked above, an immediate consequence of (1.6.3), (1.6.4), and (0.6.1) is

(1.6.5) Theorem. There is a functor $\#: \underline{\underline{AG}} \times \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$ which is a left functorial representation for the functor Mix: $(\underline{\underline{AG}} \times \underline{\underline{B}})^* \times \underline{\underline{AG}} \longrightarrow \underline{\underline{S}}$.

(1.6.6) Formulae. Let $M_i \in \underline{\underline{AG}}$, $B_i \in \underline{\underline{B}}$. Then

.1. $(M_1 \otimes M_2) \# B_1 \cong M_1 \otimes (M_2 \# B_1)$

.2. $M_1 \# (B_1 \underset{2}{\otimes} B_2) \cong (M_1 \# B_1) \# B_2$

.3. $Z \# B_1 \cong F(B_1)$ (notation of (1.6.3))

.4. $M_1 \# Z_2 \cong M_1$

.5. $M_1 \# B_1 \cong M_1 \otimes (Z \# B_1)$

.6. $Z \# (B_1 \underset{2}{\otimes} B_2) \cong (Z \# B_1) \otimes (Z \# B_2)$

.7. $(M_1 \times M_2) \# (B_1 \times B_2) = (M_1 \# B_1) \times (M_1 \# B_2) \times (M_2 \# B_1) \times (M_2 \# B_2)$

Proof: The first formula follows from the observation that there is a natural 1-1 correspondence between the $\underline{\underline{AG}}$-morphisms from either side and the functions from the direct product $M_1 \times M_2 \times B_1$ which are $\underline{\underline{AG}}$-morphisms in each of the first two variables and mixed in the last. The second formula follows from similar considerations. The third one follows from the fact that sending $f \in \mathrm{Mix}(Z, B_1; M')$ to $f_1 \in \mathrm{mix}(B_1, M')$ and sending $g \in \mathrm{mix}(B_1, M')$ to the map $f \in \mathrm{Mix}(Z, B_1; M')$ which is defined by $f(m, b) = m \cdot f(b)$ establishes a natural equivalence between the functors mix and $\mathrm{Mix}(Z, -; -)$. The proof of four is similar to the preceding: one observes that there is a natural equivalence between the functors $\mathrm{Mix}(-, Z_2; -)$ and $\underline{\underline{AG}}(-, -)$. Formula .3. and (1.6.3) have .5. as an immediate consequence, and .6. follows from .2. and .5. Finally, .7. is equivalent to the statement that both the functors $\# B : \underline{\underline{AG}} \longrightarrow \underline{\underline{AG}}$ and $M \# : \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$ send split exact sequences to split exact sequences, and that's proved like (1.4.4).

Attention should be called to the formal similarity between this step function functor and the group ring functor of elementary algebra, on the one hand, and between it and tensor product functors on the other. We suggest also that it is quite likely that the functor $\#$ is characterised by formulae .1., .2., .4., and .7. of (1.6.6), and that each of the functors $\#B : \underline{\underline{AG}} \longrightarrow \underline{\underline{AG}}$ and $M\# : \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$ preserves all exact sequences. Some of the forthcoming results will tend to support these comments.

In the meantime, observe that a ring k with a unit element can be interpreted as a category $\underline{\underline{k}}$ having only one object $*$ with $\underline{\underline{k}}(*, *) = k$ -- that is, the maps of the category $\underline{\underline{k}}$ are the elements of k, and the rule of composition is the multiplication table of k.

A category $\underline{\underline{A}}$ is <u>additive</u> if there is a functor $\underline{\underline{h}}: \underline{\underline{A}}^* \times \underline{\underline{A}} \longrightarrow \underline{\underline{AG}}$ such that $|\underline{\underline{h}}(A, B)| = \underline{\underline{A}}(A, B)$ and such that the composition rules are bilinear when lifted to $\underline{\underline{AG}}$. The prime examples, a priori, of additive categories, are the category $\underline{\underline{AG}}$ itself and the category $\underline{\underline{k}}$ associated as above to a ring k with unit element. A functor F from one additive category to another is called <u>additive</u> if, for each pair of objects A, B in the domain category, the function F_{AB} is in fact a group homomorphism from $\underline{\underline{h}}(A, B)$ to $\underline{\underline{h}}(F(A), F(B))$.

Freyd [5, p. 16] points out that the category ${}_k\underline{\underline{M}}$ of left k-modules can then be identified with the category $(\underline{\underline{k}}, \underline{\underline{AG}})_+$ of all additive functors from the category $\underline{\underline{k}}$ (associated to the ring k) to $\underline{\underline{AG}}$; indeed, such a functor consists of an object M in $\underline{\underline{AG}}$ (the image of the single object $*$ of $\underline{\underline{k}}$) and a group homomorphism $k \longrightarrow \underline{\underline{AG}}(M, M)$, which is exactly how a left k-module is defined.

Having a functor $\#: AG \times B \longrightarrow AG$, as we do, we can easily produce a functor, still denoted as $\#$,

$$(\underline{\underline{k}}, \underline{\underline{AG}})_+ \times \underline{\underline{B}} \longrightarrow (\underline{\underline{k}}, \underline{\underline{AG}})_+ :$$

namely, to the additive functor T and the boolean ring B assign the additive functor $T\#B$ which assigns to the object $*$ of $\underline{\underline{k}}$ the abelian group $T(*)\#B$ and to the map $a \in \underline{\underline{k}}(*, *) = k$ the map

$$T(a)\#id_B : T(*)\#B \longrightarrow T(*)\#B .$$

This functor $T\#B$ is easily seen to be additive, and so we are through describing the action of $\#$ on objects. The description of the behavior of $\#$ on maps is this: given a map in $(\underline{\underline{k}}, \underline{\underline{AG}})_+$, i.e., a natural transformation ϕ from an additive functor S to another T, and given a map in $\underline{\underline{B}}$, say $f \in \underline{\underline{B}}(B, B')$, define $\phi\#f : S\#B \longrightarrow T\#B'$ to be the natural transformation which assigns to the object $*$ the map $\phi(*)\#f: S(*)\#B \longrightarrow T(*)\#B'$.

Reinterpreting $(\underline{\underline{k}}, \underline{\underline{AG}})_+$ as ${}_k\underline{\underline{M}}$, we have thus obtained a functor $\#: {}_k\underline{\underline{M}} \times \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{M}}$ from whose very definition it is clear that the diagram (1.6.0) commutes (at least within natural equivalence). An equivalent way of getting to the same result is this: a k-module consists of a group M and a group homomorphism m: $k \longrightarrow \underline{\underline{AG}}(M, M)$ which converts ring multiplication to composition of maps; this group homomorphism can be reinterpreted as a group homomorphism $k \otimes M \longrightarrow M$ satisfying certain conditions, amounting to the multiplication composition condition just mentioned. Form the composite

$$k \otimes (M \# B) \cong (k \otimes M) \# B \xrightarrow{m \# \mathrm{id}_B} M \# B ;$$

it too will satisfy the conditions that assure that a k-module structure on $M \# B$ is at hand.

Formulae similar to those in (1.6.6) can now easily be established. Moreover, $\#: {}_k\underline{\underline{M}} \times \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{M}}$ is again a left functorial representation. For convenience, assume that k is commutative, and let $\otimes_k$ denote the tensor product in the autonomous category ${}_k\underline{\underline{M}}$.

(1.6.7) Theorem. Define $\mathrm{Mix}_k : ({}_k\underline{\underline{M}} \times \underline{\underline{B}})^* \times {}_k\underline{\underline{M}} \longrightarrow \underline{\underline{S}}$ as follows: $\mathrm{Mix}_k(M, B; M')$ consists of those functions $f \in \underline{\underline{S}}(M \times B, M')$ for which, in the notation of (1.6.1),

$$f_m \in \mathrm{mix}(B, M') \qquad \text{and} \qquad f^b \in {}_k\underline{\underline{M}}(M, M')$$

for all $m \in M$, $b \in B$. Then there are natural equivalences $\mathrm{mix}(B, M') \cong {}_k\underline{\underline{M}}(k \# B, M')$ and $\mathrm{Mix}_k(M, B; M') \cong {}_k\underline{\underline{M}}(M \# B, M')$, where mix is thought of as a functor from $\underline{\underline{B}}^* \times {}_k\underline{\underline{M}}$ to $\underline{\underline{S}}$, and we have formulae (meaning canonical ${}_k\underline{\underline{M}}$-isomorphisms):

.1. $(M_1 \otimes_k M_2) \# B_1 \cong M_1 \otimes_k (M_2 \# B_1)$

.2. $M_1 \# (B_1 \otimes_2 B_2) \cong (M_1 \# B_1) \# B_2$

.3. $k \# B_1 \cong k \otimes (Z \# B_1)$

.4. $M_1 \# Z_2 \cong M_1$

.5. $M_1 \# B_1 \cong M_1 \otimes_k (k \# B_1)$

.6. $k \# (B_1 \otimes_2 B_2) \cong (k \# B_1) \otimes_k (k \# B_2)$

.7. $(M_1 \times M_2) \# (B_1 \times B_2) \cong (M_1 \# B_1) \times (M_1 \# B_2) \times (M_2 \# B_1) \times (M_2 \# B_2)$

Proof: The natural equivalences are induced by those of (1.6.4) and (1.6.5). As for the formulae, .1., .2., and .4. are proved just like their counterparts in (1.6.6); .3. is an application of (1.6.6 .1.); .5. follows from .1.; .6. follows from .2. and .5.; and .7. is proved like (1.4.4).

In order to extend these results to k-algebras, we need the following

(1.6.8) Lemma. If M is a commutative k-algebra, then $M \# B$ inherits a commutative k-algebra structure.

Proof: To say M is a commutative k-algebra is to assert the presence of a ${}_k\underline{\underline{M}}$-morphism $m: M \otimes_k M \longrightarrow M$ satisfying certain conditions. These conditions are then satisfied also by the ${}_k\underline{\underline{M}}$-morphism $m': (M\#B) \otimes_k (M\#B) \longrightarrow (M\#B)$ defined to be the composite

$$(M\# B)\underset{k}{\otimes}(M\# B) \xrightarrow{\cong} ((M\# B)\underset{k}{\otimes} M)\# B \longrightarrow$$

$$\xrightarrow{\cong} (M\underset{k}{\otimes}(M\# B))\# B \xrightarrow{\cong} ((M\;\underset{k}{}\;M)\# B)\# B \longrightarrow$$

$$\xrightarrow{\cong} (M\underset{k}{\otimes} M)\# (B\underset{2}{\otimes} B) \xrightarrow{m\#\wedge} M\# B\,,$$

where $\wedge: B\underset{2}{\otimes}B \longrightarrow B$ is the multiplication -- i.e., intersection -- in the boolean ring B, and all the isomorphisms but the second, which is $tw\#\, id_B$, are provided by (1.6.7).

In terms of the following definition, we shall see that $\#$ induces a functor ${}_k\underline{\underline{A}} \times \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{A}}$ which, again, is a left functorial representation.

(1.6.9) Definition. Let M and M' be commutative rings. A function $f\colon M \longrightarrow M'$ is called

<u>multiplicative</u> if: $\quad f(ab) = f(a)f(b)$,

<u>squaremultiplicative</u>: $\quad (f(ab))^2 = f(a)f(b)$,

<u>circular</u> if: $\quad f(a+b-ab) = f(a)+f(b)-f(a)f(b)$.

With these appelations, define functors mul, sqm, cir from $\underline{\underline{B}}^* \times {}_k\underline{\underline{A}}$ to $\underline{\underline{S}}$ by assigning to the pair B, M' the set of multiplicative, squaremultiplicative, or circular functions in $\underline{\underline{S}}(B, M')$.

(1.6.10) Lemma. A function f from a boolean ring B to a commutative ring M' is mixed, multiplicative, and circular if it is any two of these things. If the ring M' is also boolean, then the mixed multiplicative functions, the mixed squaremultiplicative functions, and the boolean homomorphisms all coincide. The functor cirmul: $\underline{\underline{B}}^* \times {}_k\underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ which assigns to the pair B, M' the set of functions in $\underline{\underline{S}}$(B, M') that are at once circular and multiplicative (and hence mixed) has $k\#: \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{A}}$ as a left functorial representation, and has as right functorial representation the functor $I: {}_k\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$ which assigns to the k-algebra M the set of its idempotent elements, made a boolean ring by using circle composition (Jacobson [15, p. 8]) as union and multiplication as intersection. Proof:

The first two assertions follow from straightforward manipulation of the definitions. That $k\# B$ (which is in ${}_k\underline{\underline{A}}$ by (1.6.8)) represents the functor cirmul(B, −): ${}_k\underline{\underline{A}} \longrightarrow \underline{\underline{S}}$ follows from the fact that the equivalence of (1.6.7) between mix(B, −) and ${}_k\underline{\underline{M}}(k\# B, -)$ induces a correspondence between multiplicative functions in each mix(B, M') and ${}_k\underline{\underline{A}}$-morphisms in ${}_k\underline{\underline{M}}(k\# B, M')$. An application of (0.6.1) now shows that passage from B to $k\# B$ is a functor from B to ${}_k\underline{\underline{A}}$ and is a left functorial representation for cirmul. That I is a functor at all is due to the fact that ring homomorphisms must send idempotents to idempotents; now any function from a boolean ring to a ring which is multiplicative also sends idempotents to idempotents, and the circularity requirement is equivalent to the statement that the function sends union in the domain boolean ring to circle composition in the range ring, which is just union in the boolean ring of idempotents. This completes the proof.

Remark: This lemma shows that the functor $k\#: \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{A}}$ is left adjoint to the "boolean algebra of idempotents" functor $I: {}_k\underline{\underline{A}} \longrightarrow \underline{\underline{B}}$. We have already seen another pair of adjoint functors of this nature; namely the inclusion functor $\underline{\underline{B}} \longrightarrow {}_k\underline{\underline{A}}$ ($k = \mathbb{Z}$ or $\mathbb{Z}_2$) of (1.2.4) also has, by (0.7.7), a left adjoint, which passes, as an examination of the proof of (0.7.7) reveals, from the commutative ring or $\mathbb{Z}_2$-algebra (as the case may be) to its quotient by the equivalence relation ρ generated by the relations $a^2 \rho a$. These two adjoint pairs should not be confused, nor is a functor from one pair adjoint, on either side, to a functor from the other.

(1.6.11) Theorem. Defining a functor

$$\mathrm{Mixmul}_k : ({}_k\underline{\underline{A}} \times \underline{\underline{B}})^* \times {}_k\underline{\underline{A}} \longrightarrow \underline{\underline{S}}$$

by the requirement that $\mathrm{Mixmul}_k(M, B; M')$ be the subset of $\underline{\underline{S}}(M \times B, M')$ consisting of those functions f for which, in the notation of (1.6.1),

$$f^b \in {}_k\underline{\underline{A}}(M, M') \quad \text{and} \quad f_m \in \mathrm{mix}(B, M') \cap \mathrm{sqm}(B, M')$$

for all $b \in B$ and $m \in M$, the natural equivalence of (1.6.7) induces a natural equivalence between

$$\mathrm{Mixmul}_k(M, B; M') \quad \text{and} \quad {}_k\underline{\underline{A}}(M \# B, M'),$$

and so $\#$ is a functor ${}_k\underline{\underline{A}} \times \underline{\underline{B}} \longrightarrow {}_k\underline{\underline{A}}$ which is a left functorial representation for $Mixmul_k$, and is compatible with the earlier versions of the step function functor. The formulae of (1.6.7) are all valid in this situation, too. Moreover, the functors $\underset{2}{\otimes}$ and $\#$ when restricted to $\underline{B} \times \underline{B}$, are naturally equivalent.

<u>Proof</u>: Under the equivalence $Mix_k(M, B; M') \cong {}_{k}\underline{\underline{M}}(M\#B, M')$ of (1.6.7), the subset $Mixmul_k(M, B; M')$ of the left side is put in 1-1 correspondence with the subset ${}_{k}\underline{\underline{A}}(M\# B, M')$ of the right, which proves the first assertion. The verification of the formulae proceeds as earlier. Finally, if B and B' are two boolean rings, these formulae afford a sequence of canonical isomorphisms

$$B \underset{2}{\otimes} B' \cong B \underset{2}{\otimes} (Z_2 \# B') \cong (B \underset{2}{\otimes} Z_2) \# B' \cong B \# B' ,$$

which takes care of the last assertion and completes the proof.

(1.6.12) <u>Remark</u>. Two formulae hold in the case of k-algebras that it is meaningless to write in the other cases -- these deal with unifications and assert

$$M \# \hat{B} \cong (M \# B)^{\wedge} \qquad (M \in {}_{k}\hat{\underline{\underline{A}}})$$

$$\hat{M} \# B \cong (M \# B)^{\wedge} \qquad (B \in \hat{\underline{\underline{B}}}) .$$

The proofs consist in straightforward comparison of the unitary morphisms from each side with mixed maps that preserve unit element, and will be omitted.

One of the well known properties of tensor products of vector spaces, traditionally used in linear disjointness arguments in algebraic geometry, is that each element of the tensor product $V \otimes_k W$ of two vector spaces over the field k has an expression as $\sum_{i=1}^{n} v_i \otimes w_i$ with the v's linearly independent and the w's linearly independent, where $v \otimes w$ is the image in $V \otimes_k W$ of the pair (v, w) in $V \times W$ under the universal bilinear map $V \times W \longrightarrow V \otimes_k W$. This property has an analogue in step functions, which we now describe.

Letting M be a group and B a boolean ring, we too have a universal element $M \times B \longrightarrow M \# B$, and we may designate the image of the pair (m, b) as $m \# b$. It is sufficiently clear, at any rate, that every element of $M \# B$ has some expression as $\sum_{i=1}^{n} m_i \# b_i$, since $M \# B$ can be represented as a quotient of the free group $AG(M \times B)$ generated by $M \times B$. More is true, however.

(1.6.13) <u>Theorem</u>. If $0 \neq f \in M \# B$, then f has a unique expression

$$f = \sum_{i=1}^{n} m_i \# b_i$$

with the b's non zero and pairwise disjoint and the m's distinct and non zero.

A proof can be based inductively on the length n of the shortest possible expression for f as a sum $\sum_{i=1}^{n} m_i \# b_i$, expressing $f' = f - m_n \# b_n$ in the optimal form described by the theorem and computing $f' + m_n \# b_n = f$ from it. We shall not give the details, since we don't need this theorem, despite the fact that it has many useful applications, some of which we indicate here. The expression described in the theorem will be called the canonical expression as a sum of elementary step functions, an elementary step function being one of the elements $m\#b$ described in the prologue to the theorem.

Suppose that M is a topological group, and that B is a boolean ring. For each open neighborhood N of zero in M, define a subset U_N of $M\#B$ to consist of all elements f of $M\#B$ having, in their canonical expression as a sum of elementary step functions $f = \sum_{i=1}^{n} m_i \# b_i$, all the m'S in N. This family of subsets U_N is readily seen to form a base for the neighborhoods of zero in $M\#B$ making $M\#B$ a topological group.

If M is a normed linear space, with norm $\| \ \|$, a norm can be defined in $M\#B$ by the formula

$$\|f\| = \max_i \|m_i\|$$

where $\sum_{i=1}^{n} m_i \# b_i$ is the canonical expression for f as a sum of elementary step functions. Incidentally, the norm topology defined on $M\#B$ is the same as the topolgy described above induced by the norm topology on M.

Theorem (1.6.13) can also be used to prove that both $M\# : \underline{\underline{B}} \longrightarrow \underline{\underline{AG}}$ and $\#B : \underline{\underline{AG}} \longrightarrow \underline{\underline{AG}}$ preserve short exact sequences, for it affords an explicit description of the kernels. We shall need none of these remarks, although special cases of each of them will be established in other ways.

1.7 The Borel functor

A boolean ring (resp. σ-ring) which admits a monomorphism (resp. σ-monomorphism) to the boolean ring 2^X of all subsets of a set X is called a field (resp. σ-field) of sets. M. H. Stone has proved (in [27] and [28]) that every boolean ring is a field of sets. Precisely, the boolean ring Z_2 is an injective cogenerator (cf. (0.8.5) and the definition preceding (0.8.8)) in $\hat{\underline{\underline{B}}}$ nad has a $\underline{\underline{TD}}$-structure, where $\underline{\underline{TD}}$ is the category of compact totally disconnected Hausdorff spaces and continuous maps. That is, there is a contravariant functor $S\colon \hat{\underline{\underline{B}}} \longrightarrow \underline{\underline{TD}}$ such that $\hat{\underline{\underline{B}}}(A, Z_2) \cong |S(A)|$. Moreover, the two point discrete space Z_2, which is injective in $\underline{\underline{TD}}$ by the definition of total disconnectedness, is also a cogenerator and has a $\hat{\underline{\underline{B}}}$-structure, meaning there is a functor $R\colon \underline{\underline{TD}} \longrightarrow \hat{\underline{\underline{B}}}$ such that $|R(X)| \cong$ $\cong \underline{\underline{TD}}(X, Z_2)$ (= clopen subsets of X). Finally, S and R establish an equivalence between $(\underline{\underline{TD}})^*$ and $\hat{\underline{\underline{B}}}$. Since $S(Z_2)$ is a one point space, the category $\hat{\underline{\underline{B}}}^{!\cdot}$, which is on the one hand equivalent to $\underline{\underline{B}}$ (by (1.2.5 .2.)), is equivalent, on the other, to the category dual to the category of maps of a point to a totally disconnected compact Hausdorff space, i.e., of compact totally disconnected Hausdorff spaces with base point, and continuous base point preserving maps, call it $\underline{\underline{TD}}_*$. It follows that the boolean ring Z_2, as an object of $\underline{\underline{B}}$, has a $\underline{\underline{TD}}_*$-structure, say S_*, and the two point space Z_2 with base point 0 has a $\underline{\underline{B}}$-structure R_*, which, together with S_*, sets up an equivalence between $\underline{\underline{B}}$ and the dual of $\underline{\underline{TD}}_*$. If $B \in \underline{\underline{B}}$, we write $S(B) = S_*(B) - \{\text{base point}\}$; then $|S(B)| = \{\text{epimorphisms from } B \text{ to } Z_2\}$, and if B is unitary, no confusion arises. S(B) is called the <u>Stone space</u> of B, and $S_*(B)$, the <u>pointed Stone space</u>.

It is not the case that every σ-ring is a σ-field of sets. Indeed, an argument like that proving (1.5.4) shows that none of the σ-rings $K = 2^X/\rho$ defined just before (1.5.2) can be a σ-field of sets. Sikorski [25, (24.1)] gives necessary and sufficient conditions for a $\hat{\sigma}$-ring to be a σ-field of sets. Loomis [20] shows, however, that every σ-ring is, like these K, a quotient of a σ-field of sets by a σ-ideal, and that the Stone representation is not far off. Namely, given a σ-ring A, let I(A) be the first category (i.e., meager) subsets of S(A); then the composite

$$A \longrightarrow \left\{ \begin{matrix} \text{compact open} \\ \text{sets in } S(A) \end{matrix} \right\} \longrightarrow 2^{S(A)} \longrightarrow 2^{S(A)}/I(A)$$

where the first map is the Stone isomorphism, the second is the inclusion, and the third is the canonical projection to the quotient, is a σ-morphism, which, if A is a $\hat{\sigma}$-ring, is in fact a $\hat{\sigma}$-morphism.

In order adequately to make use of the Stone space theory, we need the notion of the quotient of a category by an equivalence relation. So let $\underline{\underline{A}}$ be an arbitrary category, and assume that there is given, for each pair A, B of objects in $\underline{\underline{A}}$, an equivalence relation ρ_{AB} on the set $\underline{\underline{A}}(A, B)$, in such a way that whenever $f_i \in \underline{\underline{A}}(A, B)$ and $g_i \in \underline{\underline{A}}(B, C)$ $(i = 1, 2)$, the implication

$$f_1 \rho_{AB} f_2, \ g_1 \rho_{BC} g_2 \Longrightarrow g_1 \cdot f_1 \rho_{AC} g_2 \cdot f_2$$

is valid. Such a class ρ of equivalence relations ρ_{AB} is admissible, or an admissible equivalence relation on $\underline{\underline{A}}$. If ρ is an admissible equivalence relation on $\underline{\underline{A}}$, there can then be formed the category $\underline{\underline{A}}/\rho$, the quotient of $\underline{\underline{A}}$ by ρ, which has the same objects as $\underline{\underline{A}}$, but the maps are given by equivalence classes of maps of $\underline{\underline{A}}$:

$$\underline{\underline{A}}/\rho(A, B) = \underline{\underline{A}}(A, B)/\rho_{AB} \quad ,$$

and the composition of two $\underline{\underline{A}}/\rho$-morphisms is, by definition, the equivalence class of the composition, in $\underline{\underline{A}}$, of any two representatives. The admissibility of ρ guarantees not only that this definition is unambiguous but that it makes $\underline{\underline{A}}/\rho$ a category. Assigning to each $\underline{\underline{A}}$-morphism f its equivalence class defines a functor, the canonical projection, from $\underline{\underline{A}}$ to $\underline{\underline{A}}/\rho$. This construction will be used in the following discussion.

A *Borel structure* on a set X is by definition a pair (B, N) consisting of a $\hat{\sigma}$-field B of subsets of X (meaning the inclusion $B \longrightarrow 2^X$ is a $\hat{\sigma}$-morphism) and a σ-ideal N of B. If (X_i, B_i, N_i) $(i = 1, 2)$ are sets with Borel structures, a function $f: X_1 \longrightarrow X_2$ is *measurable* if both

$$f^{-1}(a) \in B_1 \quad \text{whenever} \quad a \in B_2$$

and

$$f^{-1}(a) \in N_1 \quad \text{whenever} \quad a \in N_2 .$$

Thus the measurable function f induces a $\hat{\sigma}$-morphism, say $\mathbb{B}(f)$, from B_2/N_2 to B_1/N_1 by taking complete inverse images and dividing out N_1. Letting *Borel* be the category of sets with a Borel structure and measurable functions between them, sending (X, B, N) to the $\hat{\sigma}$-ring B/N and f to $\mathbb{B}(f)$ defines a contravariant functor $\mathbb{B}$: *Borel* $\longrightarrow \hat{\sigma}$. Call two measurable functions *measurably equivalent* if they yield the same $\hat{\sigma}$-morphism after applying $\mathbb{B}$; since $\mathbb{B}$ is a functor, measurable equivalence turns out to be an admissible equivalence relation, and we may form the quotient category $\overline{\text{Borel}}$ of *Borel* by this equivalence relation. Observe that $\mathbb{B}$ induces a unique contravariant functor $\overline{\mathbb{B}}$: $\overline{\text{Borel}} \longrightarrow \hat{\sigma}$ compatible with the canonical projection *Borel* $\longrightarrow \overline{\text{Borel}}$, and that $\overline{\mathbb{B}}$ is an *immersion*, by virtue of the definition of measurable equivalence. We call $\overline{\mathbb{B}}$ (or $\mathbb{B}$) the Borel functor.

Observe that Borel has all direct products. Indeed, if $i_s = (X_s, B_s, N_s)$ $(s \in S)$ is a family of object in Borel, form the direct product

$$X = \bigtimes_{s \in S} X_s ,$$

let $p_s\colon X \longrightarrow X_s$ be the canonical projections, and let (B, N) be the smallest Borel structure on X making each p_s measurable. Precisely, B is the $\hat{\sigma}$-field of subsets of X generated by all $p_s^{-1}(b_s)$ $(b_s \in B_s,\ s \in S)$, and N is the σ-ideal of B generated by all $p_s^{-1}(n_s)$ $(n_s \in N_s,\ s \in S)$. It is easy to verify that a function $g\colon Y \longrightarrow X$, where Y has a Borel structure (M, A), is measurable iff each $p_s \cdot g$ is measurable, which is enough to show that the projections p_s make (X, B, N) the direct product of the (X_s, B_s, N_s)'s. Since the canonical projection Borel $\longrightarrow$ $\overline{\text{Borel}}$ preserves direct products, $\overline{\text{Borel}}$ also has all direct products. The question arises whether the functors $\mathbb{B}$ and $\overline{\mathbb{B}}$ behave well with respect to direct products. The answer is that they have a strong tendency to convert direct products to direct sums in the category $\hat{\sigma}$, but do not always succeed in doing so. Sikorski [25, §38 A) (p. 137)] gives an example of two objects $(X, B, 0)$, $(Y, A, 0)$, for which $A \overset{\hat{\sigma}}{\oplus} B \neq \mathbb{B}((Y,A,0) \times (X,B,0))$. The tendency is realized sufficiently often for our purposes, however, as we are about to see. (The corresponding problem for direct sums in Borel has an easier and more satisfying solution which is unfortunately not needed here.)

Returning to the Stone spaces, let A be a $\hat{\sigma}$-ring, and let $B(A)$ be the $\hat{\sigma}$-field of subsets of $S(A)$ generated by the clopen subsets, let $N(A) = I(A) \cap B(A)$, and let $\Sigma(A) = (S(A), B(A), N(A)) \in$ Borel. Notice that for each $\hat{\sigma}$-morphism $f: A \longrightarrow B$, $S(f): S(B) \longrightarrow S(A)$ is in fact a measurable function from $\Sigma(B)$ to $\Sigma(A)$, which we shall call $\Sigma(f)$ in this context. From the fact that S is a contravariant functor, it follows that Σ too is a contravariant functor, from $\hat{\sigma}$ to Borel; moreover, the fact (Loomis [20]) that $A \cong B(A)/N(A)$ indicates that $\mathbb{B}\cdot\Sigma \cong \mathrm{id}_{\hat{\sigma}}$. Letting $\bar{\bar{\Sigma}}$ be the composite functor

$$\hat{\sigma} \xrightarrow{\ \Sigma\ } \text{Borel} \xrightarrow{\ \text{pr.}\ } \overline{\text{Borel}}\ ,$$

we then also have $\overline{\mathbb{B}}\cdot\bar{\bar{\Sigma}} \cong \mathrm{id}_{\hat{\sigma}}$.

An absolute Borel space is a set X equipped with a Borel structure (B, O) of such a sort that there is a Borel set Y in the countable Hilbert cube and a 1-1 correspondence between X and Y inducing an isomorphism between B and the Borel subsets of Y. We shall generally write $\mathbb{B}_X$ for B in such an instance. Let ABS (resp. $\overline{\text{ABS}}$) be the full subcategory of Borel (resp. of $\overline{\text{Borel}}$) generated by the absolute Borel spaces. It is clear that each at most countable direct product (in Borel) of absolute Borel spaces is again an anbsolute Borel space, and is the direct product in ABS. Also, the restriction of the canonical projection Borel $\longrightarrow \overline{\text{Borel}}$ to ABS yields an equivalence ABS $\longrightarrow \overline{\text{ABS}}$ in terms of which we may think of ABS as being a subcategory of $\overline{\text{Borel}}$.

Restricted to $\underline{\underline{ABS}}$, the Borel functor converts all at most countable direct products to direct sums. Precisely:

(1.7.1) Theorem. Let $A \in \hat{\sigma}$, $i \in \overline{\text{Borel}}$, $j, j_s \in \underline{\underline{ABS}}$. Then

.1. $\overline{\mathbb{B}}: \overline{\text{Borel}} \longrightarrow \hat{\sigma}$ establishes a 1-1 correspondence between $\overline{\text{Borel}}(i, j)$ and $\hat{\sigma}(\overline{\mathbb{B}}(j), \overline{\mathbb{B}}(i))$.

.2. If the index set S is at most countable,

$$\overset{\hat{\sigma}}{\bigoplus_{s \in S}} \mathbb{B}(j_s) \cong \mathbb{B}(\bigtimes_{s \in S} j_s).$$

Proof: To establish .1., it is enough, since $\overline{\mathbb{B}}$ is an immersion, to know that every $\hat{\sigma}$-morphism is obtained, a fact which is proved by Sikorski under slightly more general circumstances [25, (32.5)]. Then, using the fact that $\overline{\mathbb{B}} \cdot \overline{\Sigma} \cong id_{\hat{\sigma}}$, we obtain (using .1.)

$$\hat{\sigma}(\overset{\hat{\sigma}}{\bigoplus_{s \in S}} \mathbb{B}(j_s), A) \cong \bigtimes_{s \in S} \hat{\sigma}(\mathbb{B}(j_s), A) \cong$$

$$\cong \bigtimes_{s \in S} \hat{\sigma}(\overline{\mathbb{B}}(j_s), \overline{\mathbb{B}}(\overline{\Sigma}(A))) \cong \bigtimes_{s \in S} \overline{\text{Borel}}(\overline{\Sigma}(A), j_s)$$

$$\cong \overline{\text{Borel}}(\overline{\Sigma}(A), \bigtimes_{s \in S} j_s) \cong \hat{\sigma}(\overline{\mathbb{B}}(\bigtimes_{s \in S} j_s), \overline{\mathbb{B}}(\overline{\Sigma}(A)))$$

$$\cong \hat{\sigma}(\mathbb{B}(\bigtimes_{s \in S} j_s), A),$$

which proves .2. .

It will be convenient to call a category countably equational if it is an equational Δ-category where $\text{rank}(\Delta) \leq \aleph_0$, and an equational functor between two countably equational categories a countably equational functor.

Finally, an $\underline{\underline{A}}$-structure or an $\underline{\underline{A}}$-costructure $(Q, G, \mathcal{E})$ is called <u>countably equational</u> if $\underline{\underline{A}}$ is. Then Theorems (0.10.10) and (1.7.1) combine to indicate that $\mathbb{B}$ transfers each countably equational structure over an absolute Borel space $(X, \mathbb{B}_X, 0)$ to a countably equational costructure over the $\hat{\sigma}$-ring $\mathbb{B}_X$, in a manner compatible with countably equational functors and with $\mathrm{Str}(\underline{\underline{ABS}}, —)$-morphisms. A precise formulation is

(1.7.2) <u>Theorem</u>. For each countably equational category $\underline{\underline{A}}$, there is a unique functor

$${}_{\underline{\underline{A}}}\mathbb{B}\colon \mathrm{Str}(\underline{\underline{ABS}}, \underline{\underline{A}}) \longrightarrow \mathrm{Costr}(\hat{\sigma}, \underline{\underline{A}})$$

making the diagram

$$\begin{array}{ccc} \mathrm{Str}(\underline{\underline{ABS}}, \underline{\underline{A}}) & \xrightarrow[{}_{\underline{\underline{A}}}\mathbb{B}]{} & \mathrm{Costr}(\hat{\sigma}, \underline{\underline{A}}) \\ \downarrow & & \downarrow \\ \underline{\underline{ABS}} & \xrightarrow{\quad \mathbb{B} \quad} & \hat{\sigma} \end{array}$$

commute, and if $F\colon \underline{\underline{A}} \longrightarrow \underline{\underline{A}}'$ is a countably equational functor, then the diagram

$$\begin{array}{ccc} \mathrm{Str}(\underline{\underline{ABS}}, \underline{\underline{A}}) & \xrightarrow[{}_{\underline{\underline{A}}}\mathbb{B}]{} & \mathrm{Costr}(\hat{\sigma}, \underline{\underline{A}}) \\ {\scriptstyle \mathrm{Str}(\underline{\underline{ABS}}, F)}\downarrow & & \downarrow{\scriptstyle \mathrm{Costr}(\hat{\sigma}, F)} \\ \mathrm{Str}(\underline{\underline{ABS}}, \underline{\underline{A}}') & \xrightarrow[{}_{\underline{\underline{A}}'}\mathbb{B}]{} & \mathrm{Costr}(\hat{\sigma}, \underline{\underline{A}}') \end{array}$$

commutes.

In the following section, we present some countably equational categories to which this theorem will be applied.

1.8 Vector lattices

Definition. A lattice ordered vector algebra (resp. lattice ordered vector space) is a set V equipped with the structures both of an $\mathbb{R}$-algebra (resp $\mathbb{R}$-module) and of a lattice, provided the implications (resp. the first two implications)

(1.8.1) $x < y \Longrightarrow x + z < y + z$

(1.8.2) $x < y,\ 0 < r \in \mathbb{R} \Longrightarrow rx < ry \qquad (x, y, z \in V)$

(1.8.3) $x \leq y,\ 0 \leq z \Longrightarrow xz \leq yz$

are valid. A lattice ordered vector algebra is unitary if its underlying $\mathbb{R}$-algebra has a unit. In any lattice ordered vector space, define unary operations $^+$, $^-$, $|\ |$ by

$$x^+ = x \vee 0\ ,\quad x^- = (-x)^+\ ,\quad |x| = x^+ + x^-\ ,$$

and call two elements x and y disjoint if $|x| \wedge |y| = 0$.

(1.8.4) Remark. Conditions (1.8.1, (1.8.2), and (1.8.3) can be expressed equationally in terms of the lattice ordered vector space (algebra) operations and the additional unary operations as follows:

$$((x \wedge y) + z) \wedge (y + z) = (x \wedge y) + z\ ,$$
$$(|r|(x \wedge y)) \wedge (|r| y) = |r|(x \wedge y)\ ,$$
$$((x \wedge y)(z \vee 0)) \wedge (y(z \vee 0)) = (x \wedge y)(z \vee 0)\ .$$

Taken in conjunction with the fact that $\mathbb{R}^{\underline{M}}$, $\mathbb{R}^{\underline{A}}$, $\mathbb{R}^{\underline{\hat{A}}}$, and lattices form countably equational categories, this shows that lattice ordered vector spaces, lattice ordered vector algebras, and unitary lattice ordered vector algebras, all form countably equational categories, say $\underline{\underline{VL}}$, $\underline{\underline{AL}}$, and $\hat{\underline{\underline{AL}}}$, respectively. It can be shown, using (1.6.13), that if M is a lattice ordered vector space (algebra) and B is a boolean ring, then $M \# B$ is again a lattice ordered vector space (algebra), and that the step function functor $\#$ gives rise to functors $\underline{\underline{VL}} \times \underline{\underline{B}} \longrightarrow \underline{\underline{VL}}$, $\underline{\underline{AL}} \times \underline{\underline{B}} \longrightarrow \underline{\underline{AL}}$, $\hat{\underline{\underline{AL}}} \times \hat{\underline{\underline{B}}} \longrightarrow \hat{\underline{\underline{AL}}}$; not needing thses facts in our development, however, we shall omit their proofs.

A lattice ordered vector space is <u>boundedly σ-complete</u> (Day [4, p. 96]) or a <u>K_σ-space</u> (Vulih [32, p. 95]) if each countable family of elements $x_1, x_2, \ldots$ having an upper (lower) bound in the lattice order, has a supremum (infimum); a (unitary) lattice ordered vector algebra which is a K_σ-space is called a <u>(unitary) K_σ-algebra</u>. A lattice ordered vector space is <u>disjointedly σ-complete</u> if each countable family of pairwise disjoint positive elements has a supremum; a disjointedly σ-complete K_σ-space (resp. (unitary) K_σ-algebra) is called an <u>L_σ-space</u> (resp. <u>(unitary) L_σ-algebra</u>). K_σ- and L_σ-spaces and (unitary) K_σ- and L_σ-algebras form categories $\underline{\underline{VK}}_\sigma$, $\underline{\underline{VL}}_\sigma$, $\underline{\underline{AK}}_\sigma$, $\underline{\underline{AL}}_\sigma$, $\hat{\underline{\underline{AK}}}_\sigma$, $\hat{\underline{\underline{AL}}}_\sigma$, whose morphisms are those in $\underline{\underline{VL}}$, $\underline{\underline{AL}}$, or $\hat{\underline{\underline{AL}}}$ preserving the countable lattice operations.

(1.8.5) <u>Remark</u>. Each category $\underline{\underline{VK}}_\sigma$, $\underline{\underline{VL}}_\sigma$, $\underline{\underline{AK}}_\sigma$, $\underline{\underline{AL}}_\sigma$, $\hat{\underline{\underline{AK}}}_\sigma$, $\hat{\underline{\underline{AL}}}_\sigma$ is countably equational. Indeed, a lattice ordered vector space having an additional operation F of length ω satisfying

$$x_i \wedge F(x_0, x_1, x_2, x_3, \ldots) = x_i \wedge x_0 \qquad (i \geq 1)$$
$$x_0 \vee F(x_0, x_1, x_2, x_3, \ldots) = x_0$$
$$F(x_0, x_1 \wedge x_0, x_2 \wedge x_0, x_3 \wedge x_0, \ldots) = F(x_0, x_1, x_2, x_3, \ldots)$$
$$x \wedge F(x_0, x_1 \wedge x, x_2 \wedge x, x_3 \wedge x, \ldots) = F(x_0, x_1 \wedge x, x_2 \wedge x, x_3 \wedge x, \ldots)$$

is boundedly σ-complete, for

$$(1.8.6) \qquad F((x_i)_{i \geq 0}) = \bigvee_{i=1}^{\infty} (x_i \wedge x_0),$$

and so if $x_i \leq x_0$ $(i = 1, 2, \ldots)$, $F((x_i)_{i \geq 0}) = \bigvee_{i=1}^{\infty} x_i$, whereas if $x_i \geq x_0$ $(i = 1, 2, \ldots)$, we have $-x_i \leq -x_0$ $(i = 1, 2, \ldots)$, and $-F((-x_i)_{i \geq 0}) = -\bigvee_{i=1}^{\infty} (-x_i) = \bigwedge_{i=1}^{\infty} x_i$. In any K_σ-space, conversely, such an operation is defined exactly by (1.8.6). A K_σ-space having a second operation G of length ω satisfying

$$G((x_i)_{i \geq 0}) = G((|x_i|)_{i \geq 0})$$
$$y_i \wedge G((x_i)_{i \geq 0}) = y_i$$
$$F(G((x_i)_{i \geq 0}), y_0, y_1, y_2, \ldots) = G((x_i)_{i \geq 0})$$

where the y 's are inductively defined by

$$(1.8.7) \qquad \begin{aligned} y_0 &= |x_0| \\ y_k &= y_{k-1} \vee (|x_k| - \bigvee_{n=1}^{\infty} (|x_k| \wedge n(y_{k-1} \wedge |x_k|))) \end{aligned}$$

is an L_σ-space, for if the x 's are positive and disjoint, then $y_k = x_0 \vee \ldots \vee x_k$ by (1.8.7), and the equations that

G satisfies guarantee $G((x_i)_{i \geq 0}) = \bigvee_{i=0}^{\infty} y_i$; but $\bigvee_{i=0}^{\infty} y_i = \bigvee_{i=0}^{\infty} \bigvee_{n=0}^{i} x_n = \bigvee_{n=0}^{\infty} x_n$. In an L_σ-space, conversely, define $G((x_i)_{i \geq 0})$ as follows. Form z_i $(i \geq 0)$, which are defined by

$$z_0 = y_0 , \quad z_i = y_i - y_{i-1} \quad (i \geq 1) ,$$

where the y 's are defined by (1.8.7) -- so the z 's, apart from z_0, are just the parenthetical expression on the right side of the second formula in (1.8.7), and are therefore positive and pairwise disjoint -- and set $G((x_i)_{i \geq 0})$ equal to the union of these z 's. G obviously satisfies the first equation it should (by the very construction of the z 's), and it satisfies the other two because the union of the z 's is, by an argument similar to that on the second line of this page, identical with the union of the y 's. The exhibition of these two operations essentially proves the remark.

An element f of a lattice ordered algebra A is <u>bounded</u> if there is an idempotent $e \in A$ and a real number $n \geq 0$ such that

$$-ne \leq f \leq +ne .$$

The least such number n is called the <u>bound</u> or <u>sup norm</u> of f and is denoted $\|f\|$ (later $\|f\|_\infty$); we have

$$\|f\| \leq n \text{ iff } \exists \text{ idempotent } e \text{ with } |f| \leq ne .$$

Since $|f| \leq ne$, $|g| \leq m\bar{e}'$ imply

$$
\begin{aligned}
&|f + g| \leq (n + m)(e + e' - ee') \\
&|fg| \leq (nm)\bar{e} \\
(1.8.8)\quad &|f \vee g| \leq (n \vee m)(e + e' - ee') \\
&|f \wedge g| \leq (n \wedge m)e \quad (\text{if } f \geq 0,\ g \geq 0) \\
&|rf| \leq |r| n e \quad (r \in \mathbb{R}),
\end{aligned}
$$

we see that the set B of bounded elements in A is a lattice ordered algebra, and the inclusion of B in A is an AL-morphism. The same inequalities also indicate that B is a normed algebra. (Warning: all this is nonsense if A is not Archimedean (definition to be found, e.g., in page 81 of Vulih's book [32]); our main interest, however, lies in K_σ-algebras, which are known to be Archimedean: cf. [32, ТЕОРЕМА IV.1.5].)

Arguments like those above show that the bounded elements in a K_σ-algebra themselves form a K_σ-algebra. Moreover, if A is an L_σ-algebra (or, for that matter, a K_σ-algebra with a unit), then the idempotent elements form a boolean σ-ring and the set B of bounded elements is a Banach algebra, i.e., is sup norm complete. Indeed, if $\{e_i\}$ is a countable family of idempotents, let

$$
\begin{aligned}
&e'_0 = 0, \quad e'_i = \bigvee_{n=1}^{i} e_n \quad (i \geq 1) \\
&e''_i = e'_i - e'_{i-1} \quad (i \geq 1).
\end{aligned}
$$

Then the e''_i are disjoint positive elements, hence have a supremum $e = \bigvee_{i=1}^{\infty} e''_i$, and it is simple to check that $e = \bigvee_{i=1}^{\infty} e_i$ is an idempotent. Now let $(f_k)_{k \geq 1}$ be a Cauchy sequence in B, so that there is a sequence of positive real numbers $(t_k)_{k \geq 1}$ converging to zero such that, for each k,

$$\|f_k - f_{k+p}\| \leq t_k \quad (p \geq 1) .$$

This means there are idempotents e_{kp} such that

$$|f_k - f_{k+p}| \leq t_k e_{kp} \quad (k \geq 1 , \; p \geq 1) ;$$

let $e = \bigvee_{k=1}^{\infty} \bigvee_{p=1}^{\infty} e_{kp}$; then $|f_k - f_{k+p}| \leq t_k e$ (all k, p).

Now embed B in a σ-lattice $\overline{B}$ in such a way as to preserve the countable lattice operations (possible by (0.7.7)) and, in $\overline{B}$, form

$$h_j = \bigwedge_{k=j}^{\infty} f_k , \quad \underline{f} = \bigvee_{j=1}^{\infty} h_j = \liminf_k f_k$$

and

$$g_j = \bigvee_{k=j}^{\infty} f_k , \quad \overline{f} = \bigwedge_{j=1}^{\infty} g_j = \limsup_k f_k .$$

The relation $\underline{f} \leq \overline{f}$ follows in the usual manner from the relations $h_j \leq h_{j+1} \leq g_{j+1} \leq g_j$. From the Cauchy condition, on the other hand, we get

$$\text{(1.8.9)} \qquad \begin{aligned} \overline{f} &\leq g_{k+1} \leq f_k + t_k e \\ f_k - t_k e &\leq h_{k+1} \leq \underline{f} \end{aligned}$$

whence $\overline{f}$ and $\underline{f}$ are both in B and, since t_k converges to zero, are equal; the final fillip, that

$$\|\underline{f} - f_k\| \leq 2t_k$$

also follows from (1.8.9) and shows that B is complete.

Since an $\underline{\underline{AL}}_\sigma$-morphism sends idempotents to idempotents and preserves order and scalar multiplication, it sends bounded elements to bounded elements, in a norm-decreasing way. Hence passage from an L_σ-algebra to its (conditionally σ-complete lattice-ordered) Banach algebra of bounded elements is a functor, which we shall call

$$\text{bdd}: \underline{\underline{AL}}_\sigma \longrightarrow \mathcal{A} = \text{category of Banach algebras}.$$

(1.8.10) Lemma. The functor $\text{bdd}: \underline{\underline{AL}}_\sigma \longrightarrow \mathcal{A}$ preserves kernels and arbitrary direct products. Proof:

Let $f \in \underline{\underline{AL}}_\sigma(A, A')$, and let $x \in \ker(\text{bdd}(f))$. This means x is bounded in A and $f(x) = 0$. Say e and n are an indempotent in A and a positive real number such that $-ne \leq x \leq +ne$. We show that x is bounded in $\ker(f)$, i.e., that there is an idempotent E in $\ker(f)$ and a real number N such that $-NE \leq x \leq +NE$ (this will show that $\ker(\text{bdd}(f)) \subseteq \text{bdd}(\ker(f))$; the converse inclusion is trivial). For each positive rational number r form $E_r = e \wedge r|x|$, let $E_r^\infty = \bigwedge_{n=1,2,\ldots} (E_r)^n$, and define $E = \bigvee_{r \geq 0} E_r^\infty$. Then:

a) $f(E_r) = 0$ (clear).

b) $f(E_r^\infty) = 0$ (due to a)).

c) E_r^∞ is idempotent, so E exists, is idempotent.

d) $f(E) = 0$ (due to b)).

b) $\|x\| = N \Longrightarrow |x| \leq NE$ (proof: if $|x| \leq ne$, then $n^{-1}|x| \leq e$, hence $E_{n^{-1}} = e \wedge n^{-1}|x| = n^{-1}|x|$, and since $E \geq E_{n^{-1}}$, $nE \geq |x|$). Qed.

Next, let A_i $(i \in I)$ be a family of L_δ-algebras. Since $(e_i)_{i\in I} \in \bigtimes_{i\in I} A_i$ is idempotent if and only if each e_i is, an element $x \in \bigtimes_{i\in I} A_i$, say $x = (x_i)_{i\in I}$ is bounded if and only if each x_i is bounded <u>and</u> $\sup_{i\in I} \|x_i\| < \infty$.

Thus:

$$\mathrm{bdd}\Big(\overset{\underline{\underline{AL}}_\delta}{\bigtimes_{i\in I}} A_i\Big) = \{x \,/\, x = (x_i)_i,\ \sup_{i\in I} \|x_i\| < \infty,\ x_i \in A_i\}$$

$$= \{x \,/\, x = (x_i)_i,\ \sup_{i\in I} \|x_i\| < \infty,\ x_i \in \mathrm{bdd}(A_i)\} = \overset{A}{\bigtimes_{i\in I}} \mathrm{bdd}(A_i).$$

This completes the proof of the lemma.

(1.8.11) <u>Lemma</u>. The functor bdd preserves difference kernels and projective limits. <u>Proof</u>:

Let $f, g \in \underline{\underline{AL}}_\delta(A, A')$ and suppose $f(x) = g(x)$, with x bounded in A. We produce an idempotent E in the difference kernel of (f, g), i.e., such that $f(E) = g(E)$, and a real number n such that $|x| \leq nE$. Using the spectral theorem on the Banach algebra $B = \mathrm{bdd}(A)$, produce iterated positive square roots $|x|^{2^{-n}}$ of $|x|$ and observe that $\limsup |x|^{2^{-n}} = \liminf |x|^{2^{-n}} = E$, where E is the idempotent constructed in the previous lemma; then $f(E) = g(E)$ and $|x| \leq \|x\| E$. Thus

$$\ker(\mathrm{bdd}(f) - \mathrm{bdd}(g)) \subseteq \mathrm{bdd}(\ker(f - g)) ;$$

the converse inclusion is again obvious. For the projective limits, it need only be observed that every projective limit is the difference kernel of a suitable pair of maps between appropriately chosen direct products. Precisely, let $(P, \leq)$ be a partially ordered set (usually assumed to have the property that whenever $p, q \in P$, there is $r \in P$ satisfying $r \leq p$ and $r \leq q$), and let $F: \underline{\underline{P}} \longrightarrow \underline{\underline{A}}$ be a functor from the category $\underline{\underline{P}}$ associated to $(P, \leq)$ in (0.2.10) to an arbitrary category $\underline{\underline{A}}$; a <u>projective limit</u> of F is by definition a right representation of the contravariant functor from $\underline{\underline{A}}$ to $\underline{\underline{S}}$ sending the object $X \in \underline{\underline{A}}$ to the set

$$\{x / x = (x_p)_{p \in P} \in \bigtimes_{p \in P} \underline{\underline{A}}(X, F(p)),\ F(p \leq q) \cdot x_p = x_q\} ;$$

but for fixed X, this set of P-tuples of maps is the same as $\ker(f_X - g_X)$, where f_X and g_X are both functions from $\bigtimes_{p \in P} \underline{\underline{A}}(X, F(p))$ to $\bigtimes_{\substack{p \leq q \\ p, q \in P}} (\underline{\underline{A}}(X, F(p)) \times \underline{\underline{A}}(X, F(q)))$ defined respectively by

$$(f_X(x_p)_{p \in P})_{pq} = (x_p, x_q) ,$$

and

$$(g_X(x_p)_{p \in P})_{pq} = (x_p, F(p \leq q) \cdot x_p) .$$

Since bdd preserves products and difference kernels, therefore, it preserves projective limits, too.

For later use, we remark that there is a notion dual to that of projective limit, called <u>injective limit</u>; in brief, if a situation with respect to a category $\underline{\underline{A}}$ is such that, when transported to the dual category, it becomes a projective limit situation there, then it is an injective limit situation in the original. We may use the terms projective and inverse (resp. injective and direct) interchangeably in this context.

1.9 Step functions and functionoids

Since $(\overline{\mathbb{R}}, \mathbb{B}_{\overline{\mathbb{R}}}, 0)$ and $(\mathbb{R}, \mathbb{B}_{\mathbb{R}}, 0)$ (where $\overline{\mathbb{R}}$ denotes the two-point compactification $\{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ of the reals and $\mathbb{B}_{-}$ denotes the Borel sets of $-$) are absolute Borel spaces having a σ-lattice structure and an $\underline{\underline{AL}}_{\sigma}$-structure, respectively (indeed, the latter has an $\hat{\underline{\underline{AL}}}_{\sigma}$-structure), using the usual operations on the (extended) reals, and since the standard inclusion $\mathbb{R} \longrightarrow \overline{\mathbb{R}}$ is a measurable function compatible with the associated boundedly-σ-complete-lattice structures, it follows from Theorem (1.7.2) that the $\hat{\sigma}$-rings $\mathbb{B}_{\overline{\mathbb{R}}}$ and $\mathbb{B}_{\mathbb{R}}$ have a well defined σ-lattice costructure and $\underline{\underline{AL}}_{\sigma}$-costructure, respectively, and that, for each $\hat{\sigma}$-ring B, the inclusion $\hat{\sigma}(\mathbb{B}_{\mathbb{R}}, B) \longrightarrow \hat{\sigma}(\mathbb{B}_{\overline{\mathbb{R}}}, B)$ corresponding to the inclusion of the reals in their two-point compactification is compatible with the associated boundedly-σ-complete-lattice costructures. Much the same can be said for the costructures borne by the $\hat{\sigma}$-ring $\mathbb{B}_{\mathbb{R}^+}$ of Borel sets in the set of non negative reals, the $\hat{\sigma}$-ring $\mathbb{B}_{\mathbb{C}}$ of Borel sets in the complex numbers, the $\hat{\sigma}$-ring $\mathbb{B}_{\mathbb{T}}$ of Borel sets in the unit circle (which has an $\underline{\underline{AG}}$-costructure), etc. In particular, it can be proved that

$$\hat{\sigma}(\mathbb{B}_{\mathbb{C}}, B) \cong \mathbb{C} \underset{\mathbb{R}}{\otimes} \hat{\sigma}(\mathbb{B}_{\mathbb{R}}, B)$$

is a natural equivalence, which reduces "complex function theory" to the "real function theory" about to be presented.

By (0.9.11), it follows that $\mathbb{B}_{\mathbb{R}-\{0\}}$ has an $\underline{\underline{AL}}_\sigma$-costructure in σ, that $\mathbb{B}_{\mathbb{R}+-\{0\}}$ has a costructure, in σ, involving the operations addition, multiplication, conditional suprema of countable families, positive real scalar multiplication, all compatible with the counterpart operations involved in the $\underline{\underline{AL}}_\sigma$-costructure on $\mathbb{B}_{\mathbb{R}-\{0\}}$ -- indeed, that $\mathbb{B}_{\mathbb{R}+-\{0\}}$ is the positive cone in $\mathbb{B}_{\mathbb{R}-\{0\}}$. As the material unfolds, the meaning of these comments will become more accessible.

There are intimate relationships between the functor $\mathbb{R}\#$, the functor bdd, the $\underline{\underline{AL}}_\sigma$-costructure on $\mathbb{B}_{\mathbb{R}-\{0\}}$, and the left adjoint $L_\sigma: \underline{\underline{AK}}_\sigma \longrightarrow \underline{\underline{AL}}_\sigma$ to the inclusion functor of $\underline{\underline{AL}}_\sigma$ in $\underline{\underline{AK}}_\sigma$ (which exists by (0.7.7)), as we now reveal.

(1.9.1) <u>Theorem</u>. Viewed as functors from σ to $\dot{\mathbb{R}}^{\underline{\underline{A}}}$, there is a natural transformation

$$\mathbb{R}\# \longrightarrow \text{bdd}(\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, \text{—}))$$

making $\mathbb{R}\#B \cong$ linear span of the idempotents in $\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ for each σ-ring B; thus $\mathbb{R}\#B$ becomes a normed, lattice ordered $\mathbb{R}$-algebra, and as such it is norm-dense in $\text{bdd}(\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B))$. Finally, $\sigma(\mathbb{B}_{\mathbb{R}^{\pm}\{0\}}, \text{—})$ is naturally equivalent to $L_\sigma\cdot\text{bdd}\cdot\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, \text{—})$, i.e., to $L_\sigma\cdot$completion-in-norm$\cdot\mathbb{R}\#$. <u>Proof</u>:

The proof is preceded by a few definitions and a lemma.

An element f of $(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ is called a (real-valued) functionoid on the σ-ring B. The historical justification for this term is given by Götz [11]. If b is a Borel set in $\mathbb{R}$, the functionoid f on B is called b-valued and is said to have values in b if

$$a \in \mathbb{B}_{\mathbb{R}-\{0\}},\ a \wedge b = \emptyset \Longrightarrow f(a) = 0$$

and either

$$0 \in b$$

or

$$f(\mathbb{R}-\{0\}) \text{ is the unit element of } B.$$

The soma $f(\mathbb{R}-\{0\})$ is, in any event, called the support $\operatorname{supp}(f)$ of the functionoid f.

(1.9.2) Lemma. If f_i is a b_i-valued functionoid on the σ-ring B $(i = 1, 2, \ldots)$ and $F\colon \mathbb{R}^{\alpha} \longrightarrow \mathbb{R}$ is a measurable operation of length α $(\leq \aleph_0)$ sending the zero sequence to zero, then the induced operation, also denoted F, on $\mathfrak{I}(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ has the property that $F((f_i)_{1 \leq i < \alpha})$ is b-valued for each Borel set b in $\mathbb{R}$ containing $\{x / x = F((x_i)_{1 \leq i < \alpha}),\ x_i \in b_i\} = F((b_i)_{1 \leq i < \alpha})$.

Proof: By the theorems on the transference of structure to costructure, $F((f_i)_i)$ is the composition

$$\mathbb{B}_{\mathbb{R}-\{0\}} \xrightarrow{\mathbb{B}(F)} \mathbb{B}_{\mathbb{R}^{\alpha}-\{0\}} \cong \bigoplus_{i<\alpha}^{\sigma} \mathbb{B}_{\mathbb{R}-\{0\}} \xrightarrow{\oplus_i f_i} \bigoplus_{i<\alpha}^{\sigma} B \xrightarrow{\wedge} B ,$$

where $\mathbb{B}(F)$ sends the Borel set a to $F^{-1}(a)$, the second map is the canonical isomorphism, and the other maps are the obvious ones. It follows that if a is disjoint from a Borel set b containing $F((b_i)_i)$ then $F((f_i)_i)(a) = 0$, which proves the lemma.

(1.9.3) Corollary. A functionoid f is idempotent (a characteristic functionoid) if and only if f is $\{0, 1\}$-valued; a linear combination of idempotents (a step functionoid) if and only if it has values in a finite set; bounded if and only if it has values in some interval $[-r, +r]$ -- indeed, $\|f\| = \inf \{r / f \text{ is } [-r, +r]\text{-valued}\}$.

Proof: Each assertion is a direct consequence of the lemma.

We proceed now to the proof of the theorem. Define a function $\chi: B \longrightarrow \mathcal{S}(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ by assigning to the soma b the functionoid $\chi_b: \mathbb{B}_{\mathbb{R}-\{0\}} \longrightarrow B$ given by

$$\chi_b(a) = \begin{cases} b, & 1 \in a \\ 0, & 1 \notin a \end{cases} .$$

Observe that each χ_b is idempotent, and that χ is a mixed function, in the sense of §1.6 (also circular), i.e., that $\chi_{a \vee b} = \chi_a \vee \chi_b$ and $\chi_{a \wedge b} = \chi_a \wedge \chi_b = \chi_a \cdot \chi_b$. Writing V for the algebra of step functionoids, we now prove that χ induces an isomorphism $\mathbb{R} \# B \xrightarrow{\cong} V$.

Accordingly, let $x \in V$; say x has values in the finite set $\{r_1, r_2, \ldots, r_{n+1}\}$, where all the r_i are distinct and $r_{n+1} = 0$. By discarding and relabeling, if necessary, we may assume that $a_i = x(\{r_i\}) \neq 0 \ (1 \leq i \leq n)$, although this is not essential to the proof. In any event, the fact that x, as a homomorphism from $\mathbb{B}_{\mathbb{R}-\{0\}}$ to B, is described by

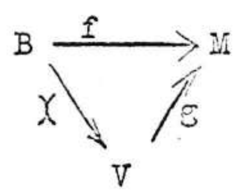
$$x(b) = \bigvee \{a_i / r_i \in b\} \qquad (b \in \mathbb{B}_{\mathbb{R}-\{0\}})$$

indicates that

$$(1.9.4) \qquad x = \sum_{i=1}^{n} r_i \chi_{a_i} \ .$$

Now let $f \in \operatorname{mix}(B, M)$ $(M \in {}_{\mathbb{R}}\underline{\underline{M}})$ and suppose that an element $g \in {}_{\mathbb{R}}\underline{\underline{M}}(V, M)$ makes the diagram

(1.9.5)



commute; then

$$(1.9.6) \qquad g(x) = \sum_{i=1}^{n} r_i f(a_i) \ .$$

Thus for each $f \in \operatorname{mix}(B, M)$ there is at most one element $g \in {}_{\mathbb{R}}\underline{\underline{M}}(V, M)$ making (1.9.5) commute; that, on the other hand, there is at least one is seen by defining $g(x)$, for $f \in \operatorname{mix}(B, M)$ and x of the form (1.9.4), by formula (1.9.6). The verification that this definition is free from contradiction is straight-forward and will be omitted, as is the proof of $\mathbb{R}$-linearity; one appeals to Lemma (1.9.2). Thus χ is a universal element making V a representation for the functor $\operatorname{mix}(B, —)$, which concludes the proof, by (1.6.7), (1.6.8), and (1.6.10), that $V \cong \mathbb{R} \# B$.

Next, let f be a bounded functionoid on B, and for each positive integer n define $f_n \in \mathbb{R}\#B$ by

$$f_n = \sum_{k=-\infty}^{+\infty} \frac{k+\frac{1}{2}}{n} \chi_{f([k/n,\ (k+1)/n))} .$$

Notice that each sum is actually finite, since the terms with $|k| \geq (n+1)\|f\|$ are zero in the sum for f_n. Since $|f - f_n| \leq n^{-1}\chi_{\mathrm{supp}(f)}$, we see that f is in the norm-closure of the step-functionoids, and so the second assertion of the theorem is verified.

Finally, if $f \in \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$, let

$$f_n = f \cdot \chi_{f([n,\ n+1))} \qquad (n \in \mathbb{Z});$$

then $|f_n| \wedge |f_k| = 0$ for $n \neq k$ and

$$f = \left(\bigvee_{n \geq 0} f_n\right) - \left(\bigwedge_{n<0} f_n\right),$$

which proves that $(\mathbb{B}_{\mathbb{R}-\{0\}}, B) = L_\sigma \cdot \mathrm{bdd} \cdot \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$, and completes the proof of the theorem, modulo the verifications of naturality, which will be omitted.

Remarks. If W is an L_σ-algebra, if B is the σ-ring of idempotents, and if $\mathrm{bdd}(W)$ is the Banach algebra of bounded elements, one can prove $L_\sigma(\mathrm{bdd}(W)) = (\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ (and $\mathrm{bdd}(W) = \mathrm{bdd}(\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B))$). Moreover, the following conditions are equivalent:

1) $W = L_\sigma(\mathrm{bdd}(W))$
2) $W = \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$

3) W has positive square roots (of positive elements)

4) for each $f \in W$ there is an idempotent $E_f \in W$ such that $|g| \wedge |f| = 0$ iff $|g| \wedge E_f = 0$

5) W has a local inversion, i.e., a unary operation sending f to ${}^{-1}f$ satisfying

a) ${}^{-1}(fg) = {}^{-1}f\ {}^{-1}g$, ${}^{-1}({}^{-1}f) = f$

b) ${}^{-1}f\ f$ is an idempotent E_f as in 4) ;

and there is a 1-1 correspondence between L_σ-algebra structures with square roots on an L_σ-space W and equivalence classes of maximal families of local order units in W (definition implicit in Kakutani [16 , Theorem 2], Vulih [32 , ГЛ. IV, §5]; called generalised weak unit by Goffman [9 , p. 112]), where a maximal family of local order units $\mathcal{F}_1$ refines another $\mathcal{F}_2$ if for each $u \in \mathcal{F}_2$ there is $v \in \mathcal{F}_1$ such that $v \wedge u = v$ and $v \wedge (u-v) = 0$ and if the union of all such v (given u) is u; two families are equivalent if they have a common refinement (under the 1-1 correspondence, the local order units become idempotents in the algebra structure). That 2) $\Longrightarrow$ 1), 3), 4), and 5) is obvious; the other statements will not be proved, not being germane to our discussion.

It is easily proved (using the "boolean algebra of idempotents" functor of (1.6.10)) that a σ-ring J is injective in the category σ if and only if $\mathrm{bdd}\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, J)$ is injective in the category of boundedly σ-complete lattice-ordered Banach algebras. The question of the existence of such injectives is as unsettled as the corresponding question for σ-rings. The related question of injective lattice ordered Banach algebras (indeed, of injective Banach spaces) is solved, in essentially the same way as the question of injective boolean rings, namely, M is an injective Banach space iff $M = \text{bdd}\,\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, J)$ with J a complete boolean ring (cf. Gleason [8], Halmos [14], and Nachbin [21]).

This is a revised and corrected version of the original §1.10, which follows later

1.10 Dense extensions

This section is devoted to the elucidation of some important properties of the largest boolean ring βA containing a given boolean ring A as a dense ideal. Call a subset A of a boolean ring B dense in B if each non zero soma of B has non zero intersection with some soma of A. If A is an ideal in B (even if merely the implication $a \in A$, $b \in B$, $b \leqslant a \Longrightarrow b \in A$ is valid), that intersection is again in A, so that an ideal is dense in B if and only if each non zero soma of B contains a non zero soma of the ideal.

(1.10.1) Lemma. Let A be an ideal in the boolean ring B. i) A is dense in B if and only if each element of B is a (perhaps infinite) union of elements of A. ii) If A is dense in B, then a boolean homomorphism defined on B is a monomorphism if and only if its restriction to A is a monomorphism.

Proof: Let A be a dense ideal in B, $b \in B$. We show

$$b = \bigvee \{a \,/\, a \leqslant b,\ a \in A\}.$$

For if an element $c \in B$ is $\geqslant a$ whenever $a \leqslant b$, we may with no loss of generality assume (replacing c by $c \wedge b$, if necessary) that $c \leqslant b$. If in fact $c \neq b$, there is a non zero $a \in A$ that is $\leqslant b \triangle c$; this element a is therefore disjoint from c, contradicting the assumptions on c; therefore $c = b$, and b is the indicated union.

Next, suppose that the restriction to A of the boolean homomorphism $f: B \longrightarrow C$ is a monomorphism. If $f(b) = 0$, then $f(a) = 0$ for each $a \leqslant b$ $(a \in A)$; consequently, the only element of A that is $\leqslant b$ is 0; by part i), $b = 0$, and f is a monomorphism.

Both converses, of course, are trivial.

By a <u>dense extension</u> of the boolean ring A we mean a monomorphism $m: A \longrightarrow B$ embedding A as a dense ideal in B. We term a dense extension <u>unifying</u> if B has a unit element. What follows is of interest only in the case that A is without unit, since each dense extension of a boolean ring with unit is easily seen, by use of (1.10.1), to be an isomorphism.

A <u>morphism</u> $m_1 \longrightarrow m_2$ between two dense extensions $m_i: A \longrightarrow B_i$ $(i = 1, 2)$ is defined to be a boolean homomorphism $f: B_1 \longrightarrow B_2$ satisfying

$$f \cdot m_1 = m_2 . \tag{1.10.2}$$

In this way we obtain a category $\mathcal{E}_A$ of dense extensions of A, containing as a subcategory the category $\mathcal{U}_A$ of unifying dense extensions of A.

(1.10.3) <u>Lemma</u>. There is never more than one morphism from one dense extension $m_1: A \longrightarrow B_1$ to another $m_2: A \longrightarrow B_2$. If there is one, it is a monomorphism, viewed as a map from B_1 to B_2. The relation $\dashrightarrow$ defined on $\mathcal{E}_A$ by $m_1 \dashrightarrow m_2$ iff there is a morphism $m_1 \longrightarrow m_2$ is a preordering with the property that if $m_1 \dashrightarrow m_2$ and $m_2 \dashrightarrow m_1$, then m_1 and m_2 are isomorphic.

<u>Proof</u>: The second assertion is a consequence of (1.10.2), (1.10.1 ii), and the fact that each m_i is a monomorphism.

Using (1.10.1 i), (1.10.2), and the now established fact that any $f: m_1 \longrightarrow m_2$ must be a monomorphism between the B_i 's, we have, for each $b \in B_1$,

$$f(b) = \bigvee \{m_2(a) / a \in A ,\ m_2(a) \leq f(b)\} =$$
$$= \bigvee \{m_2(a) / a \in A ,\ f \cdot m_1(a) \leq f(b)\} =$$
$$= \bigvee \{m_2(a) / a \in A ,\ m_1(a) \leq b\} ,$$

which establishes the first assertion. The last assertion is a consequence of the first two.

(1.10.4) <u>Lemma</u>. The following statements about a dense extension $f \in \mathcal{E}_A$ are equivalent.

i) $m \dashrightarrow f$ for all $m \in \mathcal{E}_A$;

ii) $f \in \mathcal{U}_A$ and $m \dashrightarrow f$ for all $m \in \mathcal{U}_A$.

<u>Proof</u>: This is an immediate consequence of (1.10.3) and

(1.10.5) <u>Sublemma</u>. If $m \in \mathcal{E}_A$, there is $m' \in \mathcal{U}_A$ with $m \dashrightarrow m'$. <u>Proof</u>:

Assume the dense extension $m: A \longrightarrow B$ is not unifying (otherwise take $m' = m$). Let $i_B: B \longrightarrow \hat{B}$ be the canonical injection (cf. (1.2.5)) of B in its unification. It is easy to verify that i_B is a dense extension (of B), hence $m' = i_B \cdot m: A \longrightarrow \hat{B}$ is a unifying dense extension of A , and clearly $i_B: m \longrightarrow m'$.

(1.10.6) <u>Theorem</u>. Each boolean ring A has a largest dense extension (in the sense that the equivalent conditions i), ii) of (1.10.4) are valid), say $f: A \longrightarrow \beta A$. Explicitly, βA may be taken to be the ring of compact open subsets of the Stone-Cech compactification of the Stone space of A , and $f: A \longrightarrow \beta A$ the map interpreting each compact open subset of the Stone space SA of A as a compact open set in the Stone-

Cech compactification $\beta SA \supseteq SA$ of SA.

<u>Proof</u>: We shall need to know (cf. Stone [27], [28]) that A is a dense ideal in B if and only if both i) SA is homeomorphic to the union in SB of all the compact open sets corresponding to somas of A, and ii) that union is dense in SB.

Now let $m: A \longrightarrow B$ be a dense extension. By (1.10.4) we may assume that m is unifying. Thus SA becomes a dense subset of the <u>compact</u> space SB. By the universal property of the Stone-Cech compactification, there is a (unique) continuous function $\beta SA \longrightarrow SB$ leaving SA pointwise fixed. The Stone duality converts this to a boolean homomorphism $B \longrightarrow \beta A$ which, when composed with m, yields f, qed.

(1.10.7) <u>Corollary</u>. The largest dense extension $A \longrightarrow \beta A$ may be taken to be the inclusion of the compact open subsets of SA in the ring of clopen subsets.

<u>Proof</u>: The clopen sets in SA and the compact open sets in βSA are put in one-one correspondence by assigning to each clopen set in SA its closure in βSA and to each compact open set of βSA its intersection with SA. Since the latter assignation is clearly a boolean homomorphism, while the former leaves compact open subsets of SA alone, the result is proved.

Stone [27, Def. 8] calls an ideal I in a boolean ring B <u>simple</u> if $b \in B \Longrightarrow \exists\, a \in I$ such that $b \triangle (b \wedge a)$ has zero intersection with each element of I, and proves [28, Theorem 5] that an ideal I is simple if and only if the union in SB of all the compact open sets corresponding to somas in I is clopen. Consequently, the inclusion $A \longrightarrow \beta A$ may be taken to be the inclusion of the family of all principal ideals of A in the family of all simple ideals of A. Several other representations of βA will be of use in chapter 2. We need the following lemma, however, to obtain them.

(1.10.8) Lemma. Suppose the boolean ring A appears as a subset of 2^X, for some set X. Define

$$\beta_X(A) = \{Y / Y \in 2^X,\ Y \subseteq \bigcup A,\ a \in A \Rightarrow Y \cap a \in A\}.$$

Embed A in $\beta_X(A)$ in the obvious way, and embed A in the ring ${}_{\underline{A}}\underline{M}(A, A)$ of A-module endomorphisms of A by sending $a \in A$ to the map "intersection with a". The map $\beta_X(A) \longrightarrow {}_{\underline{A}}\underline{M}(A, A)$ defined by sending $Y \in \beta_X(A)$ to the function $f_Y : A \longrightarrow A$ given by $f_Y(a) = Y \cap a$ is a boolean isomorphism, compatible with the two injections of A.

Proof: i) $\beta_X(A)$ is a boolean ring. Let $Y, Z \in \beta_X(A)$, and let & denote either symmetric difference or intersection. Whenever $a \in A$, we have

$$(Y \& Z) \cap a = (Y \cap a) \& (Z \cap a) \in A ;$$

similarly, $(Y \& Z) \cap \bigcup A = (Y \cap \bigcup A) \& (Z \cap \bigcup A) = Y \& Z$, so that $Y \& Z \in \beta_X(A)$.

ii) Each f_Y is in ${}_{\underline{A}}\underline{M}(A, A)$. This is a consequence of the formulae

$$f_Y(a \wedge a') = Y \cap a \cap a' = f_Y(a) \cap a',$$

$$f_Y(a \triangle b) = Y \cap (a \triangle b) = (Y \cap a) \triangle (Y \cap b) = f_Y(a) \triangle f_Y(b),$$

where $Y \in \beta_X(A)$ and $a, a', b \in A$.

iii) $f : Y \longrightarrow f_Y$ is a ring homomorphism. Indeed,

$$f_{Y \cap Z}(a) = Y \cap Z \cap a = f_Y(f_Z(a)),$$

and

$$f_{Y\Delta Z}(a) = (Y\Delta Z)\cap a = (Y\cap a)\Delta(Z\cap a) = f_Y(a)\Delta f_Z(a).$$

iv) f <u>is one-one</u>. For if $f_Y = 0$, then $Y\cap a = \emptyset$ for all $a\in A$, whence $\emptyset = \bigcup\{Y\cap a / a\in A\} = Y\cap \bigcup A = Y$.

v) f <u>is onto</u>. Given $g\in {}_{\underline{A}}\underline{M}(A, A)$, let $Y_g = \bigcup\{g(a)/a\in A\}$. We shall see that $Y_g\in \beta_X(A)$ and that $f_{Y_g} = g$. These facts are immediate consequences of the formulae

$$Y_g \subseteq \bigcup A,$$
$$Y_g \cap a = g(a) \quad (a\in A),$$

of which the first is obvious from the definition of Y_g. To check the second, observe first that

(1.10.9) $$g(a)\leq a \quad (a\in A) \quad (\text{since } g(a)\wedge a = g(a\wedge a) = g(a))$$

and that

$$g(a)\leq Y_g \quad (a\in A) \quad (\text{by definition of } Y_g).$$

Consequently,

$$Y_g\cap a \geq g(a).$$

The reverse inequality will follow from knowledge that

$$g(b)\cap a\leq g(a)$$

whenever $a, b\in A$. And in fact, $g(b)\cap a = g(b\cap a)$ $= g(b\cap a)\cap g(b\cap a) = g(g(b\cap a)\cap(b\cap a) = g(g(b\cap a)) = g(g(b)\wedge a)$ $= g(b)\wedge g(a)\leq g(a)$, using (1.10.9) and the fact that g is an A-module homomorphism. This completes the proof, since the compatibility assertion is trivial to verify.

<u>Remark</u>: It is a consequence of this lemma that the multiplication in ${}_{\underline{A}}\underline{M}(A, A)$, initially given as composition, is in fact determined pointwise, i.e., that $g(f(a)) = g(a)\cap f(a)$.

(1.10.10) Theorem. There are unique isomorphisms as indicated, compatible with the obvious inclusions of the boolean ring A :

i) $\beta A \cong {}_{A}\underline{\underline{M}}(A, A)$

ii) $\beta A \cong \beta_X(A)$, whenever $A \subseteq 2^X$

iii) $\beta A \cong$ inverse limit of the inverse system $(\{A_a\}_{a \in A}, \{p_{ab}: A_a \longrightarrow A_b\}_{a \geqslant b})$ of all principal ideals of A, where $p_{ab}(x) = x \wedge b$.

Proof: i) Applying (1.10.8) with $X = SA$, we see that $\beta_{SA}(A) \cong {}_{A}\underline{\underline{M}}(A, A)$. Because of (1.10.7), it remains only to show that $\beta_{SA}(A)$ consists precisely of the clopen subsets of SA. It is obvious that each clopen set is in $\beta_{SA}(A)$. Conversely, let $Y \in \beta_{SA}(A)$. Y is open because each $Y \cap a$ $(a \in A)$ is open and $Y = Y \cap \bigcup A = \bigcup \{Y \cap a \,/\, a \in A\}$. To prove Y is closed, take $x \in SA$, $x \notin Y$ (if this is impossible, Y is certainly closed, being SA). We find a neighborhood of x disjoint from Y. Take any compact open neighborhood U of x. Since $U \in A$, $U \cap Y \in A$, and in particular, $U \cap Y$ is closed. Since $x \notin U \cap Y$, the open set $U \triangle (U \cap Y)$, which is disjoint from Y, contains x.

ii) This follows from i) by an application of (1.10.8).

iii) Consider the direct system of principal ideals of A

$$(\{A_a\}_{a \in A}, \{j_{ab}: A_b \longrightarrow A_a\}_{a \geqslant b}) \quad (j_{ab}(x) = x).$$

Obviously $A = \varinjlim A_a$. Consequently, using i),

$$A \cong {}_{A}\underline{\underline{M}}(A, A) \cong {}_{A}\underline{\underline{M}}(\varinjlim A_a, A) \cong \varprojlim {}_{A}\underline{\underline{M}}(A_a, A).$$

Using (1.10.9), each A-module homomorphism from A_a to A actually takes values in A_a, so that

$$_{A}\underline{\underline{M}}(A_a\,,\ A) \cong {}_{A}\underline{\underline{M}}(A_a\,,\ A_a)\,,$$

and a simple change of rings argument shows that

$$_{A}\underline{\underline{M}}(A_a\,,\ A_a) \cong {}_{A_a}\underline{\underline{M}}(A_a\,,\ A_a)\,;$$

finally, since A_a is a boolean ring with unit,

$$_{A_a}\underline{\underline{M}}(A_a\,,\ A_a) \cong A_a\,.$$

It is left to the reader to check that the diagram

$$\begin{array}{ccc} _{A}\underline{\underline{M}}(A_a\,,\ A) & \xrightarrow{\ _{A}\underline{\underline{M}}(j_{ab}\,,\ A)\ } & _{A}\underline{\underline{M}}(A_b\,,\ A) \\ \| \wr & & \| \wr \\ A_a & \xrightarrow[\ p_{ab}\ ]{} & A_b \end{array}$$

commutes; once that is known, all is known, since the uniqueness of these three isomorphisms is due to their simple existence, by (1.10.3).

(1.10.11) Corollary. Let $\aleph$ be a cardinal number and A a boolean ring. βA is $\aleph$-complete (resp. complete) if and only if each principal ideal of A is $\aleph$-complete (resp. complete).

Proof: If βA is $\aleph$-complete, so is each principal ideal of βA, in particular, each principal ideal of A. Conversely, if each principal ideal of A is $\aleph$-complete, the fact that each map p_{ab} in the inverse limit representation of βA is a complete homomorphism guarantees that βA, the inverse limit, is also $\aleph$-complete. Take $\aleph = \text{card}\,\beta A$ for the completeness portion of the corollary.

Remark: This corollary can be used to prove that a locally compact totally disconnected Hausdorff space is basically (resp. extremely) disconnected iff its Stone-Cech compactification is (cf. [7], [29]).

Two more lemmas will be useful in chapter 2.

(1.10.12) <u>Lemma</u>. Let $\sigma: \delta \longrightarrow \sigma$ be the left adjoint to the inclusion. Then if A is a δ-ring, the canonical σ-morphism $\sigma A \longrightarrow \beta A$ induces an isomorphism $\beta A \cong \beta\sigma A$.

<u>Proof</u>: Remark first that σA is a dense extension of A , since each soma of σA is a union of at most countably many somas of A . Consequently, $\beta\sigma A$ is a dense extension of A , and as such, provides a map $\beta\sigma A \longrightarrow \beta A$, a morphism of dense extensions of A . On the other hand, the fact that σA is a dense extension of A provides a map $\sigma A \longrightarrow \beta A$, which it is not hard to see makes βA a dense extension of σA ; this is the map mentioned in the lemma. It induces a map $\beta A \longrightarrow \beta\sigma A$, which is also a morphism of dense extensions of A . By (1.10.3), the proof is complete.

(1.10.13) <u>Lemma</u>. If B is a complete boolean algebra and A is a dense ideal in B , then $B = \beta A$.

<u>Proof</u>: Since B is a dense extension of A , there is a monomorphism $g: B \longrightarrow \beta A$. Using the fact that B is injective in $\underline{\underline{B}}$ (being complete, cf. [14]), obtain a retraction $h: \beta A \longrightarrow B$ with $h \cdot g = id_B$. Both g and h are morphisms of dense extensions of A , and an application of (1.10.3) completes the proof.

(1.10.14) <u>Proposition</u>. Let $\mathcal{K}$ and $\mathcal{Y}$ be the categories of $\wedge$-complete and complete boolean rings, respectively, with complete homomorphisms (<u>not</u> necessarily preserving units). The obvious inclusion $\mathcal{Y} \longrightarrow \mathcal{K}$ has a left adjoint, sending A to βA .

<u>Remark</u>: Proposition (0.7.7) is useless in this context, since Gaifman's theorem thatt there is no free complete boolean ring on a countable set of generators (or rather, that the best candidate for such is a proper class) indicates that we are not looking for the left adjoint to an equational functor.

1.10 Dense extensions

Let $\underline{\underline{CR}}$ be the category of completely regular Hausdorff topological spaces and continuous maps, and let $\underline{\underline{K}}$ be the full subcategory generated by the compact spaces. The Stone-Čech compactification provides a functor $\beta: \underline{\underline{CR}} \longrightarrow \underline{\underline{K}}$ left adjoint to the inclusion $|\ |: \underline{\underline{K}} \longrightarrow \underline{\underline{CR}}$. Being a left adjoint, β preserves direct limits and direct sums. Moreover, $\beta \cdot |\ |$ is naturally equivalent to the identity, and it follows that direct sums and direct limits in the category $\underline{\underline{K}}$ are given, in terms of direct sums and direct limits in the category $\underline{\underline{CR}}$, by the formulæ:

$$(1.10.1) \qquad \begin{aligned} \operatorname*{dir\,lim}_{p \in P} T_p &= \beta(\operatorname*{dir\,lim}_{p \in P} |T_p|) \\ \bigoplus_{i \in I}^{\underline{\underline{K}}} T_i &= \beta(\bigoplus_{i \in I}^{\underline{\underline{CR}}} |T_i|) \end{aligned} \qquad (T_p,\ T_i \in \underline{\underline{K}}).$$

By the definition of completely regular, $\mathbb{R}$ is a cogenerator in $\underline{\underline{CR}}$, and has an $\underline{\underline{AL}}$-structure to boot, and it follows that a continuous map $f: T \longrightarrow T'$ (T, T' compact) induces an $\underline{\underline{AL}}$-isomorphism between $\underline{\underline{CR}}(T, \mathbb{R})$ and $\underline{\underline{CR}}(T', \mathbb{R})$ iff f is a homeomorphism. Since $\mathbb{R}$ behaves like an injective for the subcategory $\underline{\underline{K}}$ (in the sense of the Tietze extension theorem), each compact space T is determined by its $\underline{\underline{AL}}$-object $\underline{\underline{CR}}(T, \mathbb{R})$ (Stone-Čech classification theorem).

A completely regular space T is rather disconnected if $\underline{\underline{C}}\mathbb{R}(T, \mathbb{R})$ has enough idempotents, i.e., if for each pair of distinct points in T there is an idempotent in $\underline{\underline{C}}\mathbb{R}(T, \mathbb{R})$ separating them. To be rather disconnected, it is clearly necessary and sufficient that the clopen sets generate the topology. It follows from the definition that T is rather disconnected if and only if βT is (cf. [7, 16.D.2]), and from the topological formulation that for locally compact spaces, to be rather disconnected, to be totally disconnected, and to have a topology generated by the compact-open sets are equivalent properties: cf. [7, 16.7]. A completely regular space T is basically disconnected (resp. extremely disconnected) if $\underline{\underline{C}}\mathbb{R}(T, \mathbb{R})$ is a K_σ-algebra (resp. conditionally complete K_γ-algebra). Topologically, basic disconnectedness is equivalent to the requirement that $f^{-1}(0)$ have closed interior for each $f \in \underline{\underline{C}}\mathbb{R}(T, \mathbb{R})$, and extreme disconnectedness, to the requirement that the interior of every closed set be closed. (For other varieties of disconnectedness, see Stone [29] or Cohen [3].) Thus extreme disconnectedness implies basic disconnectedness, which in turn implies rather disconnectedness; that basic or extreme disconnectedness for a space T is is equivalent to the corresponding disconnectedness for βT follows easily from the lattice-theoretic definition. Finally, a boolean ring B is a δ-ring (resp. an $\wedge$-complete boolean ring) if and only if its Stone space (without base point) $S(B) = S_*(B) - \{\text{base pt.}\}$ is basically disconnected (resp. extremely disconnected). This follows from the embedding of B (as characteristic functions) in $\underline{\underline{C}}\underline{\underline{R}}(S(B), \mathbb{R})$ provided by the Stone isomorphism.

Call a subset A of a boolean ring B _dense in_ B if each non zero soma of B has non null intersection with some soma of A. If A is an ideal in B (even if $a' \leq a \in A \Longrightarrow a' \in A$) that intersection is again in A, so that _an ideal is dense in B iff each non zero soma of B contains a non null soma from the ideal_. A _dense extension_ of a boolean ring A is a boolean ring B containing A as a dense ideal, i.e., is a pair (B, m) with $m: A \longrightarrow B$ a monomorphism embedding A as a dense ideal in B. A _unifying extension_ of A is a dense extension (B, m) where B has a unit element. If (B_1, m_1) and (B_2, m_2) are two dense extensions of A, write $(B_1, m_1) \dashrightarrow (B_2, m_2)$ if there is a boolean homomorphism $g: B_1 \longrightarrow B_2$ such that $m_2 = g \cdot m_1$. Notice that g must be 1-1, for if $g(b_1) = 0$ with $0 \neq b_1 \in B_1$, there is $a \in A$, non zero, with $m_1(a) \leq b_1$, and it follows that $0 = g(b_1) \geq g(m_1(a)) = m_2(a) \neq 0$, a contradiction. Moreover, it can be shown that such a map g is unique, for its values must be given by the formula

$$g(b_1) = \bigvee \{m_2(a)/m_1(a) \leq b_1,\ a \in A\} \qquad (b_1 \in B_1).$$

Thus g preserves whatever unions it can.

Notice that A is a dense extension of A , and is the smallest one in the sense of the partial order $\dashrightarrow$; that the unification $(\hat{A}, i_A)$ is a dense extension of A if and only if A has no unit, and in that case is the smallest unifying extension; and that each dense extension of A is smaller than a unifying extension -- namely its unification, if it had no unit to begin with. The last observation indicates that if there is either a largest dense extension or a largest unifying extension of A , then both exist and they are equal (modulo $\dashrightarrow$). In order to prove the existence of a largest unifying extension, let us translate its universal property into the Stone space language. We shall need to know (cf. Stone [27], [28]) that A is a dense ideal in B if and only if S(A) is homeomorphic to the union in S(B) of all the compact open sets corresponding to elements of A and that union is dense in S(B) .

A unifying extension $(\beta A, j)$ of A has the property that $(B, m) \dashrightarrow (\beta A, j)$ for all unifying extensions (B, m) of A if and only if each embedding m of A as a dense ideal in a unitary boolean ring B gives rise to a unique unitary map $g: B \longrightarrow \beta A$ such that $g \cdot m = j$, which in turn is the case if and only if whenever S(A) is a dense open subset of compact S(B) there is a unique map from $S(\beta A)$ to S(B), whose restriction to S(A) is the identity, which, finally, occurs if and only if $S(\beta A) = \beta(S(A))$. This argument proves:

(1.10.2) Theorem. There is a maximal dense extension (necessarily unifying) βA for each boolean ring A, and $S(\beta(A)) = \beta(S(A))$, a fact which characterises it uniquely.

(1.10.3) Corollary. The boolean ring of clopen sets in $S(A)$ is isomorphic to βA. Proof:

The clopen sets in $S(A)$ and those in $\beta(S(A))$ are put in 1-1 correspondence by assigning to each clopen set in $S(A)$ its closure in $\beta(S(A))$ and to each clopen set in $\beta(S(A))$ its intersection with A; since the latter assignation is clearly a boolean homomorphism, the result is proved.

Stone [27, Def. 8] calls an ideal I in a boolean ring B simple if $b \in B \Rightarrow \exists\, a \in I$ such that $b \triangle (b \wedge a)$ has zero intersection with each element of I, and proves [28, Theorem 5] that an ideal I is simple if and only if the union of all the compact-open sets corresponding to elements of I is clopen. Consequently,

(1.10.4) Corollary. βA is isomorphic to the family of all simple ideals of A.

(Remark: If A has no unit then $\hat{A}$, the smallest unifying extension, has Stone space $\alpha(S(A))$, the one-point Alexandroff compactification of $S(A)$, which is just $S_*(A)$. Just as α and β are just about the only manageable compactifications in general, so are $\hat{A}$ and βA the only manageable unifying extensions.)

Observe that, in the category $\underline{\underline{CR}}$, $S(A) = \operatorname{dir\,lim}_{a \in A} S(A_a)$,

where A_a is the principal ideal $\{x / x \leq a\}$ in A generated by a. It follows that

$$S(\beta(A)) = \beta(S(A)) = \operatorname{dir\,lim}_{a \in A} S(A_a) = S(\operatorname{inv\,lim}_{a \in A} A_a),$$

using (1.10.1), where the direct limit is formed in $\underline{\underline{K}}$ and the last identification is due to the fact that S is a contravariant isomorphism. Thus:

(1.10.5) Corollary. $\beta A = \operatorname{proj\,lim}_{a \in A} A_a$.

Those who dislike Stone spaces will have no difficulty in proving (1.10.5) directly, or, if they prefer, from (1.10.6).

(1.10.6) Corollary. $\beta A = {}_{A}\underline{\underline{M}}(A, A)$.

Remark: This result was pointed out by M. Barr, who proved it directly from the definition.

Proof: The ring structure on ${}_{A}\underline{\underline{M}}(A, A)$ is of course obtained by composition; in this way ${}_{A}\underline{\underline{M}}(A, A)$ becomes an A-algebra. It is a boolean ring since

$$(f \cdot f)(a) = f(f(a)) = f(a \wedge f(a)) = f(a) \wedge f(a) = f(a)$$

for each A-morphism $f: A \longrightarrow A$ and $a \in A$. An A-algebra homomorphism from βA to ${}_{A}\underline{\underline{M}}(A, A)$ is given by sending b to the map "intersection with b". To show it's an isomorphism, take first the case when A has a unit element. Then sending $f \in {}_{A}\underline{\underline{M}}(A, A)$ to $f(1) \in A = \beta A$ provides an inverse map. In the general case, we have the isomorphisms

$$ {}_{\underline{\underline{A}}}\underline{\underline{M}}(A, A) = {}_{\underline{\underline{A}}}\underline{\underline{M}}(\operatorname*{dir\,lim}_{a \in A} A_a, A) = \operatorname*{inv\,lim}_{a \in A} {}_{\underline{\underline{A}}}\underline{\underline{M}}(A_a, A) $$

$$ \beta A = \operatorname*{inv\,lim}_{a \in A} A_a = \operatorname*{inv\,lim}_{a \in A} {}_{A_a}\underline{\underline{M}}(A_a, A_a) , $$

so we need only see that ${}_{A_a}\underline{\underline{M}}(A_a, A_a) = {}_{\underline{\underline{A}}}\underline{\underline{M}}(A_a, A)$. But

$$ f \in {}_{\underline{\underline{A}}}\underline{\underline{M}}(A_a, A) \Longrightarrow $$

$$ f(b) = b \wedge f(b) \leq b \quad (\text{all } b \leq a) \Longrightarrow $$

$$ f(b) \in A_a \text{ for } b \in A_a , $$

and so each element in ${}_{\underline{\underline{A}}}\underline{\underline{M}}(A_a, A)$ can be interpreted as an A_a-morphism from A_a to A_a. Conversely, if $f \in {}_{A_a}\underline{\underline{M}}(A_a, A_a)$, $c \in A$, and $b \in A_a$, then

$$ c \wedge f(b) = c \wedge f(b \wedge a) = c \wedge a \wedge f(b) = $$

$$ = f(b \wedge c \wedge a) = f(b \wedge a \wedge c) = f(b \wedge c) , $$

i.e., f can be interpreted as an A-morphism from A_a to A. This completes the proof of the corollary.

<u>Corollary</u> (1.10.7). A is a δ-ring if and only if βA is a σ-ring. A is a $\wedge$-complete boolean ring if and only if βA is complete. A is a σ-ring if and only if both $\beta A / A$ (the quotient in $\underline{\underline{B}}$) and βA are σ-rings.

<u>Proof</u>: A δ-ring $\Longleftrightarrow$ $S(A)$ basically disconnected $\Longleftrightarrow$ $\beta(S(A))$ basically disconnected $\Longleftrightarrow$ βA δ-ring with unit $\Longleftrightarrow$ βA σ-ring. For $\wedge$-complete replace "basically" by

"extremely" in the above argument. The last assertion of the corollary is obvious.

(1.10.8) <u>Lemma</u>. Let $\sigma : \delta \longrightarrow \sigma$ be the left adjoint to the inclusion. Then if A is a δ-ring, the canonical σ-morphism $\sigma(A) \longrightarrow \beta(A)$ induces an isomorphism

$$\beta(A) \cong \beta(\sigma(A)).$$

<u>Proof</u>: $\sigma(A)$ is a dense extension of A, since each element of $\sigma(A)$ is a union of at most countably many elements of A. Consequently, $\beta(\sigma(A))$ is a dense extension of A, and as such, comes equipped with a monomorphism $\beta(\sigma(A)) \longrightarrow \beta(A)$. On the other hand, the fact that $\sigma(A)$ is a dense extension of A indicates that there is a map from $\sigma(A)$ to $\beta(A)$, which it is not hard to see makes $\beta(A)$ a dense extension of $\sigma(A)$. This is the map mentioned in the lemma, and the map from $\beta(A)$ to $\beta(\sigma(A))$ that it induces is readily seen to be inverse to the map $\beta(\sigma(A)) \longrightarrow \beta(A)$ just constructed, which proves the lemma.

(1.10.9) <u>Lemma</u>. If B is a complete boolean algebra and A is a dense ideal in B, then $B = \beta A$. <u>Proof</u>:

Since B is a dense extension of A, there is a monomorphism $g: B \longrightarrow \beta A$. Using the fact that B is an absolute retract in $\mathcal{B}$ (being a complete boolean algebra), obtain a map $h: \beta A \longrightarrow B$ with $h \cdot g = id_B$. To see that g is an isomorphism, which will prove the lemma, it is enough to see that h, which is onto, is 1-1. Accordingly, let

$0 \neq a \in \beta A$, and suppose $h(a) = 0$. Let $j: A \longrightarrow \beta A$ be the standard inclusion, and find $a' \in A$ such that

$$0 \neq j(a') \leq a.$$

Then

$$0 = h(a) \geq h(j(a')) = h(g(a')) = a' \text{, contradiction.}$$

(1.10.10) Lemma. If $B = A/N$ (quotient in $\underline{\underline{B}}$), then $\beta B = \beta A / \tilde{\tilde{N}}$, where

$$\tilde{\tilde{N}} = \{n / n \in \beta A,\ n \wedge a \in N \text{ for all } a \in A\}.$$

False. See §2.7.

Proof: Let $f: A \longrightarrow B$ be the projection map. It induces an isomorphism $\operatorname{inv\,lim}_{b \in B} B_b \cong \operatorname{inv\,lim}_{a \in A} B_{f(a)}$.

Since $B_{f(a)} = A_a / N \cap A_a$, we see that $\beta B =$

$$= \operatorname{inv\,lim}_{a \in A} A_a / N \cap A_a = \frac{\operatorname{inv\,lim} A_a}{\{n / n \in \operatorname{inv\,lim} A_a,\ n_a \in N \text{ all } a \in A\}}$$

$$= \beta A / \tilde{\tilde{N}}.$$

(1.10.11) Lemma. Let $\mathcal{K}$ and $\mathcal{Y}$ be the categories of $\wedge$-complete and complete boolean rings respectively, with complete homomorphisms (not necessarily sending unit to unit). The obvious inclusion $\mathcal{Y} \longrightarrow \mathcal{K}$ has a left adjoint, which sends a $\wedge$-complete ring A to βA.

Remark: Proposition (0.7.7) is of no use in this situation, since Gaifman's theorem that there is no free complete boolean ring on a countable set of generators (or rather, that the one there is is a proper class) indicates that we are not dealing with the left adjoint to an equational functor.

Proof: If $\underline{\underline{m}}$ is a cardinal number, let $\underline{\underline{B}}_{(\underline{\underline{m}})}$ and $\underline{\underline{B}}_{\underline{\underline{m}}}$ be the categories of boolean rings having all $\underline{\underline{m}}$-fold intersections, resp. $\underline{\underline{m}}$-fold unions, with boolean homomorphisms preserving these operations as morphisms. By DeMorgan, each $\underline{\underline{B}}_{\underline{\underline{m}}}$-object is also a $\underline{\underline{B}}_{(\underline{\underline{m}})}$-object and each $\underline{\underline{B}}_{\underline{\underline{m}}}$-morphism likewise, so that there is a natural inclusion functor $\underline{\underline{B}}_{\underline{\underline{m}}} \longrightarrow \underline{\underline{B}}_{(\underline{\underline{m}})}$. It is an equational functor, in fact, hence has a left adjoint, say $\beta_{\underline{\underline{m}}}$. Now a boolean ring A is $\wedge$-complete if and only if A is a $\underline{\underline{B}}_{(\underline{\underline{m}})}$-object for any (hence every) $\underline{\underline{m}} \geq$ card A, and a boolean homomorphism from A to a $\wedge$-complete ring is complete if and only if it is a $\underline{\underline{B}}_{(\underline{\underline{m}})}$-morphism for any (hence every) $\underline{\underline{m}} \geq$ card A. Now card $\beta A \leq 2^{\text{card } A}$, since every element of βA is a union of elements of A, and $\underline{\underline{B}}_{\underline{\underline{m}}}(\beta_{\underline{\underline{m}}}(A), B) \cong \underline{\underline{B}}_{(\underline{\underline{m}})}(A, B)$ for each $\underline{\underline{B}}_{\underline{\underline{m}}}$-object B; in particular, if B is complete and $\underline{\underline{m}} \geq 2^{\text{card } A}$, $\gamma(\beta_{\underline{\underline{m}}}(A), B) \cong \lambda(A, B)$, by the remarks above. Finally, $\beta_{\underline{\underline{m}}}(A) = \beta(A)$, since each contains A as a dense ideal and is complete, when $\underline{\underline{m}} \geq 2^{\text{card } A}$. This proves that, on $\wedge$-complete boolean rings, β is a functor, and a rather well-behaved one at that, being a left adjoint.

We shall have occasion later to deal with what acts like its inverse. It should of course be emphasized that β is not without restriction a functor from $\underline{\underline{B}}$ to $\hat{\underline{\underline{B}}}$: only on λ does it behave well, and there, to some extent, it replaces the unification functor (l.g.o.s.).

(1.10.12) Lemma. For any δ-ring B, the norm completion $\overline{\mathbb{R}\#\beta B}$ of $\mathbb{R}\#\beta B$ in the topology induced by the functionoids on βB is isomorphic with the projective limit $\operatorname*{inv\,lim}_{b \in B} \mathbb{R}\overline{\#} B_b$. Proof:

$$\mathbb{R}\overline{\#}\beta B = \text{bdd}\,\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, \beta B) =$$
$$= \text{bdd}\,\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, \operatorname*{inv\,lim}_{b \in B} B_b)$$
$$= \text{bdd} \operatorname*{inv\,lim}_{b \in B} \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B_b)$$
$$= \operatorname*{inv\,lim}_{b \in B} \text{bdd}\,\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B_b) = \operatorname*{inv\,lim}_{b \in B} \mathbb{R}\overline{\#} B_b ,$$

using (1.8.11), (1.9.1), and (1.10.5).

Remark: For any boolean ring A, no matter what embedding of A in a σ-ring is chosen (that there is one is due to (0.7.7), as pointed out in §1.2), $\mathbb{R}\#A$ can be interpreted as the characteristic functionoids $\sum_i r_i \chi_{a_i}$ with the a_i all in A; with this identification (whose proof is like the proof of (1.9.1)), $\mathbb{R}\#A$ becomes a normed vector lattice in a unique way (the same way as indicated after (1.6.13)); we shall call its norm-completion $\mathbb{R}\overline{\#}A$, also when A fails to be a σ-ring. $\mathbb{R}\overline{\#}A$ is then always a Banach lattice (and an abstract (M)-space in the sense of Kakutani [17]).

Chapter Two

Measure and Integration

2.1 Finite measures

A real function μ defined on a boolean ring B is mixed (Definition (1.6.1)) if and only if whenever $a \wedge b = 0$ the equation $\mu(a \vee b) = \mu(a) + \mu(b)$ is valid, i.e., if and only if μ is a finite measure, in the traditional terminology to which we now revert. We thus know from §1.6 that the finite measures on B and the linear transformations $\mathbb{R} \# B \longrightarrow \mathbb{R}$ are in natural 1-1 correspondence. If i_μ denotes the transformation corresponding to a finite measure μ, we have the following easily verified properties.

(2.1.1) i_μ is a positive linear functional iff $\mu(a) \geq 0$ for all $a \in B$;

(2.1.2) $(i_\mu)^+ = i_{\mu^+}$, where $\mu^+(a) = \sup\{\mu(b) / b \leq a\}$;

(2.1.3) $(i_\mu)^- = i_{\mu^-}$, where $\mu^-(a) = \sup\{-\mu(b) / b \leq a\}$;

(2.1.4) If μ is positive, i_μ is bounded if and only if $\sup\{\mu(b) / b \in B\}$ is finite, and in either case $\|i_\mu\| = \sup\{\mu(b) / b \in B\}$;

(2.1.5) For any finite measure μ, i_μ is bounded if and only if $\|\mu\|_1 = \sup\{\mu^+(b) / b \in B\} + \sup\{\mu^-(b) / b \in B\}$ is finite, and in either case, $\|i_\mu\| = \|\mu\|_1$.

(2.1.6) $i_\mu = r\, i_{\mu_1} + s\, i_{\mu_2}$ iff $\mu(a) = r\mu_1(a) + s\mu_2(a)$ for all $a \in B$.

Let $V_1(B)$ denote the normed vector lattice of finite measures μ on B for which $\|\mu\|_1$ is finite.

(2.1.7) Theorem. The natural equivalence between finite measures on B and linear transformations from $\mathbb{R}\#B$ to $\mathbb{R}$ induces an order preserving, isometric, linear isomorphisms between $V_1(B)$ and the continuous linear functionals on the norm completeion $\mathbb{R}\bar{\#}B$ of $\mathbb{R}\#B$. Proof:

$\mathbb{R}\bar{\#}B$ is intended as in the remark following (1.10.12). Properties (2.1.1) through (2.1.6) indicate that $V_1(B)$ and the continuous linear functionals on $\mathbb{R}\#B$ are in a 1-1 correspondence of the typ desired by the theorem; but the continuous linear functionals on $\mathbb{R}\#B$ have unique extensions to the completion, which proves the theorem.

For the rest of this section, and in the next section, we shall drop the adjective finite when speaking about finite measures. The adjective will be reinstated in §2.3.

Suppose now that B is a σ-ring. A measure μ on B is a σ-measure, or countably additive, if it satisfies

(2.1.8) $$a_i \wedge a_j = 0 \quad (i \neq j) \Longrightarrow \mu\Big(\bigvee_{i=1}^{\infty} a_i\Big) = \sum_{i=1}^{\infty} \mu(a_i) \quad .$$

(It is intended that the series converge absolutely, i.e., unconditionally.)

(2.1.9) Lemma. If μ is a σ-measure on a σ-ring B, then μ^+ and μ^- are also σ-measures, and $\mu \in V_1(B)$.

Proof: If $a_i \wedge a_j = 0$ $(i \neq j)$ and $a = \bigvee_{i=1}^{\infty} a_i$, we show $\mu^+(a) = \sum_{i=1}^{\infty} \mu^+(a_i)$. For each i, choose $a_{in} \leq a_i$

such that $\mu(a_{in}) \geq \mu^+(a_i) - \frac{1}{n}$ $(n = 1, 2, \ldots)$, and let $b_i = \bigvee_{n=1}^{\infty} a_{in}$; an easy argument shows $\mu(b_i) = \mu^+(a_i)$, and so $\sum_i \mu^+(a_i) = \sum_i \mu(b_i) = \mu(\bigvee_i b_i) \leq \mu^+(a)$. On the other hand, selecting $b_n \leq a$ so that $\mu(b_n) \geq \mu^+(a) - \frac{1}{n}$ $(n = 1, 2, \ldots)$ and setting $b = \bigvee_n b_n$, we obtain $\mu(b) = \mu^+(a)$. Now let $b'_i = b \wedge a_i$. We then see

$$\mu^+(a) = \mu(b) = \sum_i \mu(b'_i) \leq \sum_i \mu^+(b'_i) \leq \sum_i \mu^+(a_i),$$

which proves μ^+ is a σ-measure. The argument for μ^- is similar. Now suppose that somas c_n can be found with $\mu^+(c_n) \geq n$ $(n = 1, 2, \ldots)$. Then $\mu^+(\bigvee_n a_n) \geq n$, for each integer n, and so μ^+ is not a finite measure, which is a contradiction; hence μ^+ is bounded; similarly, μ^- is bounded. It follows that $\mu \in V_1(B)$, and the lemma is proved.

Write $W_1(B)$ for the subspace of $V_1(B)$ consisting of the σ-measures on B, where B is a σ-ring. Let μ be a positive element of $V_1(B)$, B still a σ-ring, and suppose that the extension I_μ to $\mathbb{R}\overline{\#}B = \mathrm{bdd}\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ of the linear functional i_μ on $\mathbb{R}\# B$ has the Daniell (or Beppo-Levi) property:

$$(2.1.10) \quad f_n \in \mathbb{R}\overline{\#}B,\ f = \bigvee_n f_n \in \mathbb{R}\overline{\#}B,\ f_n \leq f_{n+1} \Longrightarrow I_\mu(f_n) \longrightarrow I_\mu(f).$$

Then if $a_i \wedge a_j = 0$, $a_i \in B$ $(i = 1, 2, \ldots)$, we have

$$\sum_{i=1}^{n} \mu(a_i) = \sum_{i=1}^{n} I_\mu(\chi_{a_i}) = I_\mu(\sum_{i=1}^{n} \chi_{a_i}) = I_\mu(\chi_{\bigvee_{i=1}^{n} a_i});$$

the Daniell property ensures that $I_\mu(\chi_{\bigvee_{i=1}^{n} a_i})$ converges to $I_\mu(\chi_{\bigvee_{i=1}^{\infty} a_i}) = \mu(\bigvee_{i=1}^{\infty} a_i)$, so that if I_μ is positive, bounded, and Daniell, μ is a σ-measure. The converse is true, that if μ is a positive σ-measure, then I_μ is Daniell, so that the space $W_1(B)$ corresponds, under the isomorphism of (2.1.7), to the linear combinations of positive bounded Daniell functionals on $\mathbb{R} \# B$. The proof is not immediate, however, but requires some preparatory material. In the meantime, we may call a continuous linear functional on $\mathbb{R} \# B$ <u>Daniell</u> if its positive and negative parts have the Daniell property.

(2.1.11) <u>Lemma</u>. B is a σ-ring, μ is a positive σ-measure on B, f and f_k $(k = 1, 2, \ldots)$ are real functionoids on B (not necessarily bounded), with $\limsup_k f_k = f = \liminf_k f_k$. For each pair of real numbers $e > 0$, $d > 0$, there is a soma $B(e, d)$ and an integer N such that $\mu(B(e, d)) < e$, and such that whenever X is a soma disjoint from $B(e, d)$, the inequality $\|\chi_X (f_k - f)\| \leq d$ is valid for all $k \geq N$.

<u>Proof</u>: Subtracting f from each f_k, if necessary, it may be assumed, with no loss of generality, that $f = 0$. Restricting attention to the principal ideal generated by

any soma containing the union of the supports of all the f_k, if necessary, we may assume that B is a $\hat{\sigma}$-ring, with no loss of generality, since if X is disjoint from the support of each f_k, it is disjoint also from the support of f, and consequently $\| \chi_X (f_k - f) \| = 0$. The functionoids f, f_k may therefore be thought of as $\hat{\sigma}$-morphisms from $\mathbb{B}_{\mathbb{R}}$ to B, and the assumption f = 0 means that for each Borel set b in $\mathbb{R}$,

$$f(b) = \begin{cases} 0, & 0 \notin b \\ 1, & 0 \in b, \end{cases}$$

where 1 denotes the unit element of B. Letting

$$\begin{aligned} V_k &= \bigvee_{n=k}^{\infty} f_n((-\infty, -d)) \\ U_k &= \bigwedge_{n=k}^{\infty} f_n((-\infty, +d)) \end{aligned} \qquad (k = 1, 2, \ldots)$$

the hypothesis $\liminf_k f_k = 0 = \limsup_k f_k$ indicates that

$$\bigvee_{k=1}^{\infty} U_k = 1, \quad \bigwedge_{k=1}^{\infty} V_k = 0,$$

$$U_1 \leq \cdots \leq U_{n-1} \leq U_n, \quad V_n \leq V_{n-1} \leq \cdots \leq V_1.$$

Hence, introducing the notation

$$A_k = U_k \triangle (U_k \wedge V_k), \quad B_k = 1 \triangle A_k,$$

we have

$$A_1 \leq \cdots \leq A_{k-1} \leq A_k, \quad B_k \leq B_{k-1} \leq \cdots \leq B_1,$$

$$\begin{aligned} \bigvee_{k=1}^{\infty} A_k &= \bigvee_{k=1}^{\infty} (U_k \triangle (U_k \wedge V_k) = \bigvee_{k=1}^{\infty} U_k \triangle \bigwedge_{k=1}^{\infty} (U_k \wedge V_k) \\ &= 1 \triangle 0 = 1, \end{aligned}$$

$$\bigwedge_{k=1}^{\infty} B_k = 0,$$

and the fact that μ is a σ-measure implies

$$\lim_k \mu(A_k) = \mu(1), \quad \lim_k \mu(B_k) = 0 .$$

Choose N so that $\mu(B_N) < e$, and let $B_N = B(e, d)$. The inclusions

$$A_N = 1 \triangle B_N \leq U_N \leq f_k((-\infty, +d))$$
$$A_N \wedge f_k((-\infty, -d)) \leq A_N \wedge V_N = 0 \qquad (\text{all } k \geq N)$$

indicate that $\chi_{A_N} f_k$ is $[-d, +d]$-valued $(k \geq N)$, i.e., that $\| \chi_{1 \triangle B_N} f_k \| \leq d$ whenever $k \geq N$, which proves the lemma.

(2.1.12) Corollary: Egoroff's Theorem. In the situation of the previous lemma, there is a soma $B(e)$, for each real number $e > 0$, outside of which the convergence is uniform and for which $\mu(B(e)) < e$. Proof:

Use the lemma to construct somas $B_n = B(2^{-n}e, \frac{1}{n})$ and integers N_n corresponding to the pairs $(2^{-n}e, \frac{1}{n})$, and let $B(e) = \bigvee_{n=1}^{\infty} B_n$. Since $\mu(B_n) < 2^{-n}e$, we have $(B(e)) < \sum_{n=1}^{\infty} 2^{-n} e = e$. To see that the convergence is uniform outside $B(e)$, let $d > 0$ be given. Choose an integer $n \geq d^{-1}$, and let $k \geq N_n$. Then if the soma X is disjoint from $B(e)$, $\| \chi_X (f_k - f) \| \leq \frac{1}{n} \leq d$ (by the lemma, since $X \wedge B_n = 0$), and the corollary is proved.

(2.1.13) Corollary. If μ is a positive σ-measure on a σ-ring B, then I_μ is Daniell. Proof:

It is enough to verify that if $(f_k)_{k=1,2,\ldots}$ is a decreasing sequence of bounded, positive functionoids with infimum zero, then $I_\mu(f_n)$ converges to zero. To this end, letting $e > 0$ be given, we produce an integer N so that $I_\mu(f_k) \leq e$ whenever $k \geq N$. First of all, obtain a soma $B = B(\frac{e}{2\|f_1\|})$, using the Egoroff Theorem, so that whenever $X \wedge B = 0$, $\| \chi_X f_n \|$ converges to zero uniformly with respect to X. Then

$$0 \leq I_\mu(f_n \chi_B) \leq \|f_n\| \mu(B) \leq \|f_1\| \mu(B) < \frac{e}{2} ,$$

since $0 \leq f_n \chi_B \leq \|f_n\| \chi_B \leq \|f_1\| \chi_B$, for all integers n. Therefore, selecting N so that whenever $k \geq N$ and $X \wedge B = 0$, $\|f_k \chi_X\| \leq \frac{e}{2\|\mu\|_1}$, and letting $A = \bigvee_{n=1}^{\infty} \operatorname{supp}(f_n)$, we see that whenever $k \geq N$, we get the inequalities

$$\begin{aligned} I_\mu(f_k) &= I_\mu(\chi_A f_k) = I_\mu(\chi_{B \wedge A} f_k) + I_\mu(\chi_{A \triangle (B \wedge A)} f_k) \\ &\leq I\ (\chi_B f_k) + I_\mu(\chi_{A \triangle (B \wedge A)} f_k) \\ &\leq \frac{e}{2} + \frac{e}{2\|\mu\|_1} \mu(A \triangle (B \wedge A)) \leq \frac{2e}{2} = e , \end{aligned}$$

which completes the proof, both of the corollary, and of

(2.1.14) Theorem. B is a σ-ring. Under the isometry between $V_1(B)$ and the dual space $(\mathbb{R} \# B)^*$, the space $W_1(B)$ and the Daniell functionals are put in 1-1 correspondence with each other.

A finite measure μ on a complete boolean ring is called <u>normal</u> or <u>completely additive</u> if it satisfies

$$(2.1.15) \quad a_i \wedge a_j = 0 \ (i \neq j) \Longrightarrow \mu(\bigvee_{i \in I} a_i) = \sum_{i \in I} \mu(a_i)$$

for each index set I, where the sum is intended to converge as a net over the finite subsets of I. The proof that μ^+ and μ^- are normal if μ is can be modeled on the proof for σ-measures, once it is observed that at most a countable number of somas in any disjoint family can have non zero measure (otherwise the sum in (2.1.15) wouldn't exist).

It must be pointed out that if B is complete, then $\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ is conditionally complete in its lattice order. For if $(f_i)_{i \in I}$ is a family of positive functionoids, define a functionoid $f \in \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ by specifying

$$(2.1.16) \quad \begin{aligned} f((-\infty, 0)) &= 0 \\ f((0, r)) &= \bigvee_{i \in I} f_i((0, r)) \qquad (r > 0) . \end{aligned}$$

Since $\bigvee_{j=1}^{\infty} f((0, r_j)) = \bigvee_{j=1}^{\infty} \bigvee_{i \in I} f_i((0, r_j)) =$

$\bigvee_{i \in I} \bigvee_{j=1}^{\infty} f_i((0, r_j)) = \bigvee_{i \in I} f_i((0, \sup_j r_j)) = f((0, \sup_j r_j))$,

there is a unique σ-morphism $f: \mathbb{B}_{\mathbb{R}-\{0\}} \longrightarrow B$ with $f((-\infty, 0))$ and $f((0, r))$ as specified in (2.1.16), and f is obviously the infimum of the family $(f_i)_{i \in I}$.

A few more observations are in order.

(2.1.17) .1. If B is a σ-ring (resp. complete boolean ring) and μ is any positive extended-real-valued function on B satisfying $\mu(0) = 0$ and

(2.1.18) $$b \leq \bigvee_{i \in I} a_i \Longrightarrow \mu(b) \leq \sum_{i \in I} \mu(a_i)$$

for each countable (resp. arbitrary) index set I, then

$$N_\mu = \mu^{-1}(0) = \{a \,/\, a \in B\,,\ \mu(a) = 0\}$$

is a σ-ideal (resp. complete ideal) in B.

.2. A positive σ-measure μ on a complete boolean ring is normal iff N_μ is a complete ideal.

.3. If μ is a positive σ-measure on the σ-ring B and $0 \leq f \in \mathbb{R} \# B$, then $I_\mu(f) = 0$ if and only if $\mathrm{supp}(f) \in N_\mu$.

.4. If B is a complete boolean ring (resp. a σ-ring) and $(f_i)_{i \in I}$ is a family (resp. countable family) of functionoids whose supremum $f = \bigvee_{i \in I} f_i$ exists, then $\mathrm{supp}(f) \leq \bigvee_{i \in I} \mathrm{supp}(f_i)$; moreover, if $f_i \geq 0$ and $f \overset{\%}{\in} \mathbb{R} \# B$, then $I_\mu(f_i) = 0 \Longrightarrow I_\mu(f) = 0$ if μ is a positive normal measure (resp. positive σ-measure).

Proof: .1. That N_μ is non void follows from $\mu(0) = 0$. That $\bigvee_{i \in I} a_i \in N_\mu$ if $a_i \in N_\mu$ $(i \in I)$ and that $b \in N_\mu$ if $b \leq a \in N_\mu$ both follow directly from (2.1.18).

.2. Assume N_μ is complete and let n be its maximal element, $n = \bigvee\{b \,/\, b \in N_\mu\} \in N_\mu$. Then $\mu(b) = 0$

if and only if $b \leq n$ and $[0 \neq b \leq a \Longrightarrow \mu(b) \neq 0]$ if and only if $a \wedge n = 0$. Hence if $(a_i)_{i \in I}$ is a disjoint family of elements in B, writing $b_i = a_i \triangle (a_i \wedge n)$, we have

$$b_i \wedge b_j = 0 \ (i \neq j), \quad a_i \wedge n \wedge b_i = 0,$$
$$\mu(a_i) = \mu(a_i \wedge n) + \mu(b_i) = \mu(b_i),$$

at most countably many b_i are $\neq 0$.

It follows that $\mu(\bigvee_i a_i) = \mu(\bigvee_i (b_i \vee (a_i \wedge n))) = \mu(\bigvee_i b_i \vee \bigvee_i (a_i \wedge n)) = \mu(\bigvee_i b_i) + \mu(\bigvee_i (a_i \wedge n)) = \sum_i \mu(b_i) = \sum_i \mu(a_i)$, and μ is normal. The converse is contained in .1. .

.3. If $\mathrm{supp}(f) \in N_\mu$, the relations

$$0 \leq f \leq \|f\| \chi_{\mathrm{supp}(f)}, \quad I_\mu(\chi_{\mathrm{supp}(f)}) = \mu(\mathrm{supp}(f))$$

show that $I_\mu(f) = 0$, since I_μ preserves order, μ being positive. Conversely, writing

$$a_n = f\left(\left(\frac{\|f\|}{n+1}, \frac{\|f\|}{n}\right]\right),$$

the relations

$$a_n \wedge a_{n+1} = 0, \quad \bigvee_n a_n = \mathrm{supp}(f), \quad 0 \leq \frac{\|f\|}{n+1} \chi_{a_n} \leq f \chi_{a_n} \leq f$$

yield (if $f \neq 0$, otherwise there's nothing to prove):

$$0 \leq \mu(\mathrm{supp}(f)) = \sum_n \mu(a_n) = \sum_n I_\mu(\chi_{a_n}) =$$
$$= \sum_n \frac{n+1}{\|f\|} I_\mu\left(\frac{\|f\|}{n+1} \chi_a\right) \leq \sum_n \frac{n+1}{\|f\|} I_\mu(f),$$

so that if $I_\mu(f) = 0$, then $\mathrm{supp}(f) \in N_\mu$.

.4. The relation among the supports is obvious from (2.1.16); making use of .3. and .1., it implies the subsequent statement.

If μ is a bounded positive measure on a complete boolean ring B (so that $\mathbb{R}\#B = \mathrm{bdd}\,\phi(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$ is conditionally complete (cf. (2.1.16))), and if I_μ is <u>completely</u> <u>additive</u> in the sense that

$$\text{(2.1.19)}\quad \left.\begin{array}{l} f, f_i \in \mathbb{R}\#B \\ f_i \wedge f_j = 0 \ (i \neq j) \\ f = \bigvee_{i\in I} f_i \end{array}\right\} \Longrightarrow I_\mu(f) = \sum_{i\in I} I_\mu(f_i)$$

for any index set I, then using the relations

$$\bigvee_i \chi_{a_i} = \chi_{\bigvee_i a_i}, \quad \mu(a) = I_\mu(\chi_a),$$

μ is easily proved to be normal. For the converse, suppose that μ is normal: then in particular, μ is a σ-measure and I_μ is Daniell. Let f_i, f be as in (2.1.19). Since then $\mathrm{supp}(f_i) \wedge \mathrm{supp}(f_j) = 0 \ (i \neq j)$,

$$0 \leq \sum_i I_\mu(f_i) \leq \sum_i I_\mu(\|f\|\, \chi_{\mathrm{supp}(f_i)}) = \sum_i (\|f\|\, \mu(\mathrm{supp}(f_i))$$
$$= \|f\| \sum_i \mu(\mathrm{supp}(f_i)) = \|f\|\, \mu(\bigvee_i \mathrm{supp}(f_i)) < \infty ;$$

consequently the set $J = \{i \,/\, I_\mu(f_i) > 0\}$ is at most countable, and by (2.1.17 .4.) we obtain $I_\mu(\bigvee_{i\in I-J} f_i) = 0$ and

$$\sum_{i\in I} I_\mu(f_i) = \sum_{i\in J} I_\mu(f_i) = I_\mu(\bigvee_{i\in J} f_i) = I_\mu(\bigvee_{i\in J} f_i) + I_\mu(\bigvee_{i\in I-J} f_i)$$
$$= I_\mu(\bigvee_{i\in J} f_i + \bigvee_{i\in I-J} f_i) = I_\mu(\bigvee_{i\in I} f_i) .$$

Thus I_μ is completely additive, and we have proved

(2.1.20) Theorem. Under the canonical isomorphism $V_1(B) \cong (\mathbb{R}\overline{\#}B)^*$, there arises (if B is a complete boolean ring) a 1-1 correspondence between the space $L_1(B)$ of normal measures and the completely additive continuous linear functionals (= linear combinations of positive such).

Remark: That $V_1(B)$ is complete in its norm $\| \|_1$ is obvious from the fact that it's a dual space. That $L_1(B)$ and $W_1(B)$ are complete (i.e., closed subspaces of $V_1(B)$) when B is complete, resp. σ-complete, can be proved directly, but follows trivially from the results of the next section.

2.2 Two canonical projections

For each σ-ring B, we shall produce a projection $T(B): V_1(B) \longrightarrow W_1(B)$, and for each complete boolean ring, a projection $W_1(B) \xrightarrow{R(B)} L_1(B)$. These projections will be uniquely determined by a few simple requirements in terms of which the reader will have no difficulty in proving that T and R are natural transformations between the contravariant functors

$$\sigma \longrightarrow \underline{\underline{B}} \xrightarrow{V_1} \text{normed vector lattices}$$

and

$$\sigma \xrightarrow{W_1} \text{normed vector lattices} ,$$

and between

$$\mathcal{Y} \longrightarrow \sigma \xrightarrow{W_1} \text{normed vector lattices}$$

and

$$\mathcal{Y} \xrightarrow{L_1} \text{normed vector lattices} ,$$

respectively. The results and methods of this section are directly inspired by [10].

Let B be a complete boolean ring. Define

$$(2.2.1) \qquad \begin{aligned} E_B &= \{b \,/\, b \in B,\ B_b \text{ has a strictly positive } \sigma\text{-measure}\} \\ &= \{b \,/\, b \in B,\ B_b \text{ has a strictly positive normal measure}\} \end{aligned}$$

(here B_b, as usual, denotes the principal ideal $\{a \,/\, a \leq b\}$ in B generated by b, and a measure μ on a ring A is *strictly positive* if

$$(2.2.2) \qquad a \neq 0 \Longrightarrow \mu(a) > 0).$$

That a strictly positive σ-measure is normal (which proves the definition, in a manner of speaking) follows from (2.1.17 .2.).

(2.2.3) Lemma. E_B is a σ-ideal in B. Proof:

If $b \leq a \in E_B$, the restriction to $B_b \subseteq B_a$ of any strictly positive normal measure on B_a is a strictly positive normal measure on B_b. If $a_k \in E_B$ $(k = 1, 2, \ldots)$, let $b_k = a_k \triangle \bigvee_{n=1}^{k-1} (a_n \wedge a_k)$. Then

$$b_i \wedge b_j = 0 \ (i \neq j), \quad b_k \leq a_k, \text{ and } \bigvee_{k=1}^{\infty} a_k = \bigvee_{k=1}^{\infty} b_k .$$

Letting μ_k be a strictly positive measure on B_{b_k}, it is readily checked that the formula

$$\mu(a) = \sum_{k=1}^{\infty} \frac{\mu_k(a \wedge b_k)}{2^k \|\mu_k\|_1} \qquad (a \leq \vee_k a_k)$$

defines a strictly positive σ-measure on the principal ideal generated by $\bigvee_k a_k$. This completes the proof.

The inclusion $E_B \subseteq B$ defines a restriction map $f \colon W_1(B) \longrightarrow W_1(E_B)$, which is obviously a bounded, order-preserving linear transformation of norm ≤ 1.

(2.2.4) Lemma. Each normal measure on B is uniquely dötermined by its restriction to E_B. Proof:

If μ and λ are normal measures on B, and if their restrictions to E_B are identical, the restriction of their difference $\mu - \lambda$ to E_B is identically zero.

Let $e = \bigvee \{b / b \in E_B\}$. Every normal measure on B vanishes outside e (i.e., on somas disjoint from e). To prove this, let η by a positive normal measure on B; let n_η be the maximal element in the complete ideal N_η; and let $b_\eta = 1 \triangle n_\eta$ (1 is the unit in the complete boolean ring B). The restriction of η to B_{b_η} is strictly positive, hence $b_\eta \in E_B$, hence $b_\eta \leq e$, and so if $x \wedge e = 0$, then $x \wedge b_\eta = 0$ and $\eta(x) = 0$. For an arbitrary normal measure, an elementary argument with positive and negative variations establishes the result.

Now $\mu - \lambda$ (returning to the proof of the lemma) being normal, vanishes on somas disjoint from e. On the other hand, being normal and vanishing on E_B, it vanishes also on the principal ideal generated by e, and therefore vanishes on all of B, i.e., $\mu = \lambda$.

As a converse result, we have

(2.2.5) Lemma. Every element of $W_1(E_B)$ is the restriction of a normal measure on B. Proof:

If μ is a positive σ-measure on E_B, its boundedness assures the existence of a soma $b \in E_B$ outside which it vanishes. So extend μ by zero to all of B, i.e., define $\tilde{\mu}$ on B by the formula

$$\tilde{\mu}(a) = \mu(a \wedge b) \qquad (a \in B).$$

That $\tilde{\mu}$ is a normal measure on B follows from the facts that the projection $B \longrightarrow B_b$ is a complete boolean homomorphism and that the restriction of μ to B_b is (by default) a normal measure on B_b (since the existence on B_b of some strictly positive normal measure ensures that each disjoint family of non null subsomas of b is at most countable). That the restriction of $\tilde{\mu}$ to E_B is μ is clear.

We are now in a position to prove the first main result of this section, in which B is assumed to be complete.

(2.2.6) <u>Theorem</u>. There is a linear transformation $R: W_1(B) \longrightarrow W_1(B)$ having the properties

.1. $R^2 = R$;

.2. $R(\mu) = \mu$ iff μ is normal;

.3. $0 \leq \mu \implies 0 \leq R(\mu) \leq \mu$.

These characterise R uniquely, and, unless $R = 0$, R is bounded with norm 1 . <u>Proof</u>:

<u>Uniqueness</u>. Any projection (condition .1.) is uniquely determined by its range (condition .2.) and nullspace; by means of the three conditions, we show that the nullspace of R must be such that its positive elements (which of course determine it completely) are those σ-measures μ for which $0 \leq \lambda \leq \mu$, λ normal, implies $\lambda = 0$. Indeed, if μ is a σ-measure (positive) for which the implication holds, the fact that $R(\mu)$ is normal (.2. and .1.) together with the fact that $0 \leq R(\mu) \leq \mu$ shows $R(\mu) = 0$. On the other hand, if μ is in the kernel of R and is positive,

and if $0 \leq \lambda \leq \mu$, with λ normal, the relations

$$0 \leq \lambda = R(\lambda) \leq R(\mu) = 0$$

show that $\lambda = 0$. This proves the uniqueness.

Boundedness. If $R \neq 0$, condition .3. assures that R is bounded with norm at most 1 , and condition .1. assures that the norm is at least 1 .

Existence. By (2.2.4) and (2.2.5), the composite

$$L_1(B) \xrightarrow[\text{incl.}]{g} W_1(B) \xrightarrow[\text{restr.}]{f} W_1(E_B)$$

is an isomorphism ϕ . Define R to be the composite

$$W_1(B) \xrightarrow{f} W_1(E_B) \xrightarrow{\phi^{-1}} L_1(B) \xrightarrow{g} W_1(B) .$$

Since $f \cdot g = \phi$, we have

$$R^2 = g \cdot \phi^{-1} \cdot f \cdot g \cdot \phi^{-1} \cdot f = g \cdot \phi^{-1} \cdot \phi \cdot \phi^{-1} \cdot f = g \cdot \phi^{-1} \cdot f = R ,$$
$$R \cdot g = g \cdot \phi^{-1} \cdot f \cdot g = g \cdot \phi^{-1} \cdot \phi = g ,$$

which establish conditions .1. and .2., respectively. That R sends positive measures to positive measures follows from the evident fact that f , g , and ϕ do; finally, if $\mu \geq 0$, that $R(\mu) \leq \mu$ follows from the fact that $f(\mu)$ vanishes outside some soma $b_\mu \in E_B$ for which $R(\mu)(a) = \mu(a \wedge b_\mu)$, and the last number is not greater than $\mu(a)$, qed.

Remark: The proof could equally well have been accomplished using linear functionals in place of measures; as an indication, we prove the next theorem by means of them.

(2.2.7) Theorem. If B is a σ-ring, there is a linear transformation $T: V_1(B) \longrightarrow V_1(B)$ having the properties

.1. $T^2 = T$;

.2. $T(\mu) = \mu$ iff μ is a σ-measure;

.3. $0 \leq \mu \implies 0 \leq T(\mu) \leq \mu$.

These characterise T uniquely, and, unless $T = 0$, T is bounded with norm 1. Proof:

The uniqueness and norm arguments are negligeable variations of the corresponding parts of the proof of (2.2.6), and will therefore be omitted. For the existence proof, observe that (2.1.7) and (2.1.14) allow us to work in $(\mathbb{R} \bar{\#} B)^*$; in other words, we shall produce a linear transformation $T: (\mathbb{R}\bar{\#} B)^* \longrightarrow (\mathbb{R}\bar{\#} B)^*$ which is a projection, leaves a functional alone iff it is Daniell, and, together with $Id_{(\mathbb{R}\bar{\#}B)^*} - T$, sends positive functionals to positive functionals. The arguments here more closely resemble those of [10] than those used in the proof of (2.2.6).

Let f be a positive continuous linear functional on $\mathbb{R}\bar{\#}B$. Define two classes of sequences by

$$\mathbb{K}(f) = \{(f_n)_{n=1,2,\ldots} / 0 \leq f_n \leq f_{n+1} \ , \ f = \bigvee_n f_n\}$$
$$\mathbb{L}(f) = \{(g_n)_{n=1,2,\ldots} / 0 \leq g_n \ , \ \sum_n g_n = f\} ,$$

where by $\sum_n g_n$ is meant the supremeum over all partial sums. Next, define functions between these classes as follows:

$$F_1 : \mathbb{K}(f) \longrightarrow \mathbb{L}(f)$$
$$F_2 : \mathbb{L}(f) \longrightarrow \mathbb{K}(f)$$
$$F_3 : \mathbb{K}(f+g) \longrightarrow \mathbb{K}(f) \times \mathbb{K}(g)$$
$$F_4 : \mathbb{K}(f) \times \mathbb{K}(g) \longrightarrow \mathbb{K}(f+g)$$
$$F_5 : \mathbb{K}(f) \longrightarrow \mathbb{K}(r f) \qquad (r \geq 0)$$

are defined by

$$F_1((f_n)_n) = (g_n)_n \ , \quad \text{where} \quad g_n = \begin{cases} f_1 \ , & n = 1 \\ f_n - f_{n-1} \ , & n \ \ 1 \end{cases} ;$$

$$F_2((g_n)_n) = (f_n)_n \ , \quad \text{where} \quad f_n = \sum_{k=1}^{n} g_k \ ;$$

$$F_3((h_n)_n) = ((f_n)_n \, , (g_n)_n) \, , \quad \text{where} \quad f_n = f \wedge h_n \, , \ g_n = h_n - f_n \ ;$$

$$F_4((f_n)_n \, , (g_n)_n) = (h_n)_n \ , \quad \text{where} \quad h_n = f_n + g_n \ ;$$

$$F_5((f_n)_n) = (r f_n)_n \ .$$

If F is a positive linear functional in $(\mathbb{R} \overline{\#} B)^*$, define

$$\left.\begin{aligned} T(F)(f) &= \operatorname{glb} \{ \lim_n F(f_n) \,/\, (f_n)_n \in \mathbb{K}(f) \} \\ T'(F)(f) &= \operatorname{glb} \{ \textstyle\sum_n F(g_n) \,/\, (g_n)_n \in \mathbb{L}(f) \} \end{aligned}\right\} \qquad (0 \leq f \in \mathbb{R} \overline{\#} B).$$

Simple calculations with F_1 and F_2 show that $T(F)(f)$ and $T'(F)(f)$ are the same. Using F_3 and F_4 one obtains $T(F)(f + g) = T(F)(f) + T(F)(g)$, when f and g are both positive. Finally, by use of F_5, it can be seen that $T(F)(r f) = r\, T(F)(f)$ $(0 \leq r \in \mathbb{R}$, f positive$)$.
It being clear from the definition of $T(F)$ that

$$(2.2.8) \qquad 0 \leq F \in (\mathbb{R} \overline{\#} B)^*, \ 0 \leq f \in \mathbb{R} \overline{\#} B \implies 0 \leq T(F)(f) \leq F(f) \, ,$$

it follows that $T(F)$ has a unique extension to a (continuous) positive linear functional on $\mathbb{R}\overline{\#}B$. It is also clear from the definition of $T(F)$ that $T(F) = F$ if and only if F is Daniell (for positive F).

The same argument, essentially, that proves $T(F)$ is positive homogeneous (using F_5) proves that T is positive homogeneous, as a permutation of the positive elements in $(\mathbb{R}\overline{\#}B)^*$. Next, let F and G be two positive elements of $(\mathbb{R}\overline{\#}B)^*$; we show $T(F+G) = T(F) + T(G)$ by proving that, for each positive element $f \in \mathbb{R}\overline{\#}B$, $T(F+G)(f) = T(F)(f) + T(G)(f)$. Accordingly, let such a functionoid f be given, as well as a real number $\varepsilon > 0$. Choose $(f_n)_n \in \mathbb{K}(f)$ so that

$$\lim_n \langle\!\langle (F+G)(f_n) \rangle\!\rangle \leq T(F+G)(f) + \varepsilon .$$

It follows that

$$T(F)(f) + T(G)(f) \leq \lim_n F(f_n) + \lim_n G(f_n) =$$
$$= \lim_n \langle\!\langle (F+G)(f_n) \rangle\!\rangle \leq T(F+G)(f) + \varepsilon ,$$

and, the inequality between the extremes above being available for each $\varepsilon > 0$, we have

$$T(F)(f) + T(G)(f) \leq T(F+G)(f) .$$

Conversely, given $\varepsilon > 0$ once again, let $(f_n)_n, (g_n)_n \in \mathbb{K}(f)$ be two sequences for which

$$\lim_n F(f_n) \leq T(F)(f) + \frac{\varepsilon}{2} ,$$
$$\lim_n G(g_n) \leq T(G)(f) + \frac{\varepsilon}{2} .$$

Then, setting $h_n = f_n \wedge g_n$, $(h_n)_n \in \mathbb{K}(f)$ and

$$T(F+G)(f) \leq \lim_n (F+G)(h_n) = \lim_n F(h_n) + \lim_n G(h_n) \leq$$
$$\leq \lim_n F(f_n) + \lim_n G(g_n) \leq T(F)(f) + T(G)(f) + \frac{2\varepsilon}{2},$$

so that $T(F+G)(f) \leq T(F)(f) + T(G)(f) + \varepsilon$, for each ε; together with the previous inequality, this shows

$$T(F+G)(f) = T(F)(f) + T(G)(f)$$

for each positive f, and consequently, $T(F+G) = T(F) + T(G)$.

Thus T has a unique extension to a (continuous, but that's irrelevant) positive linear transformation from $(\mathbb{R} \# B)^*$ to itself, and $T(F) = F$ if and only if F is Daniell -- this was remarked before for positive F, and it follows by linearity now for all F: this takes care of condition .2. in Theorem (2.2.7), while (2.2.8) takes care of condition .3. . It remains only to prove that T is a projection.

To this end, let $0 \leq f \in \mathbb{R} \# B$, let $0 \leq F \in (\mathbb{R} \# B)^*$, and let a real number $\varepsilon > 0$ be given. If $(f_n)_n \in \mathbb{L}(f)$ satisfies

$$\sum_n T(F)(f_n) \leq T(T(F))(f) + \frac{\varepsilon}{2},$$

choose $(g_{nm})_m \in \mathbb{L}(f_n)$ $(n = 1, 2, \ldots)$ such that

$$\sum_m F(g_{nm}) \leq T(F)(f_n) + \frac{\varepsilon}{2^{n+1}} \qquad (n = 1, 2, \ldots).$$

Then

$$T(F)(f) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F(g_{nm}) \leq \sum_{n=1}^{\infty} (T(F)(f_n) + \frac{\varepsilon}{2^{n+1}}) =$$

$$= \sum_{n=1}^{\infty} T(F)(f_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} \leq T(T(F))(f) + \frac{2\varepsilon}{2} .$$

Since the inequality between the extremes above holds for all ε, $T(F)(f) \leq T^2(F)(f)$; the converse inequality follows from (2.2.8), and so $T(F)(f) = T^2(F)(f)$, for positive, hence for all, functionoids $f \in \mathbb{R}\overline{\#}B$. Consequently, $T^2(F) = T(F)$ for positive, hence for all, linear functionals $F \in (\mathbb{R}\overline{\#}B)^*$, and T is a projection. This completes the proof of (2.2.7).

(2.2.9) Corollary. If B is a σ-ring (resp. a complete boolean ring), then $W_1(B)$ (resp. $L_1(B)$) is complete in its norm $\| \|_1$ induced from $V_1(B)$.

Proof: $V_1(B) = (\mathbb{R}\overline{\#}B)^*$ is obviously complete, and $W_1(B)$ (resp. $L_1(B)$) occurs as the nullspace of the continuous linear transformation $Id - T$ (resp. $Id - R$) hance is a closed subspace of $V_1(B)$ and is therefore complete (resp. is a closed subspace of $W_1(B)$ and is therefore complete).

2.3 Non finite measures

The measures to be considered from now on need not be finite, as they were in the preceding sections; precisely, a measure on a boolean ring B is an extended-real-valued function $\mu : B \longrightarrow \bar{\mathbb{R}}$ satisfying

$$(2.3.1) \qquad \mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b), \quad \text{and} \quad \mu(0) = 0 .$$

It is assumed that each side of the first equation is always defined, i.e., that μ omits at least one of the infinite values $\pm\infty$. Defining the positive and negative variations μ^+ and μ^- of μ by the formulae occurring on the right sides of (2.1.2) and (2.1.3), respectively, at least one of these variations is a finite measure, and conversely, if η and λ are positive measures, one of which is finite, then $\eta - \lambda$ (defined by $(\eta-\lambda)(a) = \eta(a) - \lambda(a)$) is a measure.

A measure μ on a σ-ring is a σ-measure if the implication

$$(2.3.2) \qquad a_i \wedge a_j = 0 \ (i \neq j) \Longrightarrow \mu(\bigvee_{i \in I} a_i) = \sum_i \mu(a_i)$$

is valid for all countable families of somas (it is intended that the series converge absolutely, perhaps to $\pm\infty$). If μ is a σ-measure, so are μ^+ and μ^-; conversely, the difference of two positive σ-measures, one of which is finite, is demonstrably a σ-measure. The details will be left to the reader. A measure on a complete boolean ring is normal if the implication (2.3.2) is valid for any family of somas; similar remarks about the variations are valid, as for σ-measures.

The measures with which we deal will, for the most part, be either finite or positive, since any measure is a difference of such and our results will be reasonably additive.

For any positive measure μ on a boolean ring B , define

$$N_\mu = \mu^{-1}(0) = \{b / b \in B,\ \mu(b) = 0\},$$
$$F_\mu = \{a / a \in B,\ \mu(a) < +\infty\}.$$

N_μ is clearly an ideal in F_μ , which is, in turn, an ideal in B . Moreover, the restriction of μ to F_μ is a finite measure there, hence defines a positive linear functional i_μ on $\mathbb{R}\# F_\mu$ (cf. Remarks following (1.10.12) or (1.6.13)), in terms of which pseudonorms $\| \|_{p,\mu}$ can be defined on $\mathbb{R}\# F_\mu$ $(1 \leq p \in \mathbb{R})$ by the formula

$$\| f \|_{p,\mu} = (i_\mu(|f|^p))^{p^{-1}},$$

in which $|f|^p$ is intended to mean the step function $\sum_{i=1}^{n} |r_i|^p \# a_i$ if $f = \sum_{i=1}^{n} r_i \# a_i$ $(a_i \wedge a_j = 0,\ i \neq j)$. The completion of $\mathbb{R}\# F_\mu$ in the pseudonorm $\| \|_{p,\mu}$ is called $L_p(\mu)$ and is, of course, Banach, indeed, is an L_p-space in the sense of [Bohnenblust, Duke Math. J. 6 (1940), pp. 627-640] . This construction has a modicum of naturality, which will be described but not stressed. Namely, if $\phi \in \underline{\underline{B}}(A, B)$ and μ is a measure on B , then $\mu \cdot \phi$ is a measure on A , ϕ sends somas in $F_{\mu \cdot \phi}$ to F_μ , and $\mathbb{R}\#\phi : \mathbb{R}\# F_{\mu\cdot\phi} \longrightarrow \mathbb{R}\# F_{\mu}$ induces an order-preserving isometry from $L_p(\mu\cdot\phi)$ to $L_p(\mu)$. (It is tempting to write $L_\infty(\mu)$ for one of the spaces $\mathbb{R}\overline{\#} F_\mu$, $\mathbb{R}\overline{\#} B$, $\mathbb{R}\overline{\#}(F_\mu / N_\mu)$, or $\mathbb{R}\overline{\#}(B/N_\mu)$; this temptation is, however, incompatible with the desire to assert that $(L_1(\mu))^* = L_\infty(\mu)$. Thorp [30] gives examples to this effect; we shall give the right definition.)

In the event that μ is a positive measure on a δ-ring B whose restriction to each principal ideal is a σ-measure there, there is a unique σ-measure $\sigma\mu$ on σB satisfying

$$(\sigma\mu)(b') = \mu(b') \qquad (b' \in B),$$

where σB is the σ-ring associated to B by the left adjoint to the inclusion functor $\sigma \longrightarrow \delta$ and is thought of as containing B. Indeed, each element of σB is an at most countable union of elements of B, and $\sigma\mu$ is described by

$$(\sigma\mu)(b) = \inf \{ \textstyle\sum_i \mu(b_i) / (b_i)_{i=1,2,\dots} \subseteq B, \bigvee_i b_i = b \}.$$

Since the δ-rings F_μ and $F_{\sigma\mu}$ stand in the relation

$$F_\mu \subseteq F_{\sigma\mu} \subseteq \sigma F_\mu ,$$

we have an isomorphism $\sigma F_\mu \cong \sigma F_{\sigma\mu}$, and it follows by a norm-density argument that $L_p(\mu) \cong L_p(\sigma\mu)$. Thus, the behavior of such measures which, locally, are σ-measures is already revealed in the study of σ-measures.

If μ is a positive σ-measure on a σ-ring B , define

$$E_\mu = \sigma F_\mu \ ,$$
$$B_\mu = \beta E_\mu \ ,$$
$$\tilde{N}_\mu = \{ n / n \in B_\mu \ , \ n \wedge b \in N_\mu \text{ for all } b \in E_\mu \} \ .$$

Then, using (1.10.8) and (1.10.10), we obtain

$$\beta(E_\mu / N_\mu) = B_\mu / \tilde{N}_\mu \ ,$$
$$B_\mu = \beta F_\mu \ ,$$
$$\tilde{N}_\mu = \{ n / n \in B_\mu \ , \ n \wedge b \in N_\mu \text{ for all } b \in F_\mu \} \ ,$$
$$\beta(E_\mu / N_\mu) = \beta(F_\mu / N_\mu) \ .$$

Define $L_\infty(\mu) = \text{bdd}\,\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, \beta(E_\mu / N_\mu)) = \mathbb{R}\overline{\#}\,\beta(E_\mu / N_\mu)$; the norm here will henceforth be denoted by $\| \ \|_\infty$. That this is the right definition, in the sense that $L_1{}^* = L_\infty$, will be proved in §2.5 , by means of the Radon-Nikodym theorem and an elementary direct and inverse limit argument. The Radon-Nikodym theorem needs some machinery, however, and it is this we now construct.

Let μ be a positive σ-measure on a σ-ring B . A σ-measure λ on B is called μ-<u>normal</u> if

.1. $N_\mu \subseteq N_\lambda$ (λ is <u>absolutely</u> <u>continuous</u> with respect to μ , $\lambda \ll \mu$) ;

.2. there is a soma $e \in E_\mu$ such that $a \in N_\lambda$ if $a \wedge e = 0$ (e is a <u>carrier</u> of λ) ;

.3. $E_\mu \subseteq F_\lambda$;

.4. λ and μ are comparable over e .

Notice that λ is in fact a finite measure by virtue of .2. and .3. . To define comparability, we first define a Hahn decomposition of e for λ with respect to μ : this consists of a real number t and disjoint somas e_t^+, e_t^- whose union is e and which satisfy

$$a \leq e_t^+ \Longrightarrow \lambda(a) \leq t\,\mu(a) ,$$
$$a \leq e_t^- \Longrightarrow \lambda(a) \geq t\,\mu(a) .$$

λ and μ are comparable over e if the parameters t involved in the available Hahn decompositions of e for λ with respect to μ are dense in the space of real numbers.

If condition .3. in the above definition is replaced by

.3a. $E_\mu \subseteq E_\lambda$,

we say λ is almost μ-normal. By means of the fundamental theorem of calculus, which characterises the positive almost μ-normal measures, we obtain a description of $L_1(\mu)$ as a special version of the Radon-Nikodym theorem. We need one more set of definitions, however, before we can proceed.

If B is a δ-ring, let $B^o = B - \{0\}$ be the set of non null somas of B ; for $b \in B^o$ and $f \in S(\mathbb{B}_{\mathbb{R}-\{0\}}, B)$, define $\inf_b(f) = m_f(b)$ and $\sup_b(f) = M_f(b)$ thus:

$$\text{(2.3.3)} \quad \begin{aligned} m_f(b) &= \sup \{ r / r\,\chi_b \leq f\,\chi_b \ , \ r \in \mathbb{R} \} \\ M_f(b) &= \inf \{ r / r\,\chi_b \geq f\,\chi_b \ , \ r \in \mathbb{R} \} \end{aligned}$$

(the null set convention is $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = +\infty$).

In view of the fact that $r\chi_b \le f\chi_b$ if and only if $-r\chi_b \ge -f\chi_b$, we see at once that

$$(2.3.4) \quad m_f(b) = -M_{-f}(b) \qquad (b \in B^o).$$

As functions from B^o to $\overline{\mathbb{R}}$, m_f and M_f have the following properties:

$$(2.3.5) \quad a \le b \implies m_f(b) \le m_f(a) \le M_f(a) \le M_f(b);$$

$$(2.3.6) \quad b = \bigvee_{i \in I} b_i \implies m_f(b) = \inf_i m_f(b_i),\ M_f(b) = \sup_i M_f(b_i);$$

$$(2.3.7) \quad b = \bigwedge_{i \in I} b_i \implies m_f(b) = \sup_i m_f(b_i),\ M_f(b) = \inf_i M_f(b_i);$$

$$(2.3.8) \quad M_f(b) = \sup\{m_f(a) / 0 \ne a \le b\};$$

$$(2.3.9) \quad m_f(b) = \inf\{M_f(a) / 0 \ne a \le b\};$$

$$(2.3.10) \quad f \ge 0 \iff m_f(b) \ge 0 \text{ for all } b \in B^o;$$

$$(2.3.11) \quad f = 0 \iff m_f(b) = 0 \text{ (all } b) \iff M_f(b) = 0 \text{ (all } b);$$

(2.3.12) f is bounded $\iff m_f(\mathrm{supp}(f))$ and $M_f(\mathrm{supp}(f))$ are finite; moreover $\|f\|_\infty = \max\{|m_f(\mathrm{supp}\, f)|, |M_f(\mathrm{supp}\, f)|\}$;

$$(2.3.13) \quad f = \bigwedge_{i \in I} f_i \implies m_f(b) = \inf_i m_{f_i}(b);$$

$$(2.3.14) \quad f = \bigvee_{i \in I} f_i \implies M_f(b) = \sup_i M_{f_i}(b).$$

<u>Proof</u>: Observe to begin with that $m_f(b)\chi_b \le f\chi_b$ if $m_f(b)$ is finite, while if $m_f(b) = -\infty$, there is still a strong sense in which the above inequality is meaningful. Similarly for $M_f(b)$. We shall, for the most part, therefore, fail to distinguish the finite from the infinite cases in the proofs of these properties.

If $a \leq b$, then $a = b \wedge a$ and so $m_f(b)\chi_a = m_f(b)\chi_{b\wedge a}$ $= m_f(b)\chi_b\chi_a \leq f\chi_b\chi_a = f\chi_a$; hence $m_f(b) \leq m_f(a)$. Now, using (2.3.4), $M_f(a) = -m_{-f}(a) \leq -m_{-f}(b) = M_f(b)$. Finally, $m_f(a)\chi_a \leq f\chi_a \leq M_f\chi_a$ shows $m_f(a) \leq M_f(a)$, and completes the proof of (2.3.5).

To prove (2.3.6), observe that $r\chi_{b_i} \geq f\chi_{b_i}$ for all i if and only if $r\chi_b \geq f\chi_b$: this proves the second identity, and the first is now a consequence of (2.3.4).

A similar argument proves (2.3.7).

To rpove (2.3.8), notice first that the sup is $\leq M_f(b)$, according to (2.3.5). If the sup differs from $+\infty$ (otherwise we're through), call it S. We distinguish three cases.

i) $0 \neq b \leq \mathrm{supp}(f)$. Then b has non null intersection with some soma $f([n, n+1] - \{0\}) = a_n$ $(n \in Z)$ and $m_f(a_n \wedge b) \geq n$ for such a soma: hence $S \neq -\infty$. If $S \neq M_f(B)$, let $\varepsilon = M_f(b) - S > 0$, and for each integer n, define

$$b_n = f([M_f(b) - n^{-1}, M_f(b)] - \{0\}) \qquad (n \geq 1).$$

If $n > \frac{1}{\varepsilon}$, then $b_n \wedge b \neq 0$, and so $m_f(b \wedge b_n) \geq m_f(b_n) \geq$ $\geq M_f(b) - n^{-1}$, which proves the sup is $\geq M_f(b)$ and establishes the formula.

ii) $0 \neq b$, $b \wedge \mathrm{supp}(f) = 0$. Then $m_f(b) = 0 = M_f(b)$.

iii) $b = c \vee d$, $0 \neq c \leq \mathrm{supp}(f)$, $d \neq 0 = d \wedge \mathrm{supp}(f)$. This case follows from the previous cases by use of (2.3.6). This concludes the proof of (2.3.8).

(2.3.9) follows from (2.3.8) by use of (2.3.4).

(2.3.10, 11, and 12) all follow from the easily verified facts that $m_f(b) = m_{\chi_b f}(b)$, $M_f(b) = M_{\chi_b f}(b)$, and that f , qua σ-morphism from $\mathbb{B}_{\mathbb{R}-\{0\}}$ to B , vanishes on each Borel set disjoint from $C - \{-\infty, 0, +\infty\}$ (C is a closed connected set in $\overline{\mathbb{R}}$) if and only if $m_f(\mathrm{supp}(f))$ and $M_f(\mathrm{supp}(f))$ are both in C. (This fact can also be used to advantage in the proof of (2.3.8).)

To prove (2.3.13), observe, as for (2.3.6), that $r\chi_b \leq f\chi_b$ iff for each i, $r\chi_b \leq f_i\chi_b$.

By an application of (2.3.4), (2.3.14) follows from (2.3.13).

2.4 The Fundamental Theorem of Integral Calculus

This section is almost entirely devoted to the proof of a result embracing the mean value theorem, the Radon-Nikodym theorem, the chain rule, and the change-of-variables formula. In what follows, $\mathbb{R}^+ = \{t / 0 < t \in \mathbb{R}\}$ denotes the positive reals. The σ-ring $\mathbb{B}_{\mathbb{R}+}$ of course inherits from $\mathbb{B}_{\mathbb{R}-\{0\}}$ a costructure involving addition, multiplication, lattice, and scalar operations in a fairly evident manner.

(2.4.1) Theorem. Let μ be a positive σ-measure on a σ-ring B. For each $f \in \sigma(\mathbb{B}_{\mathbb{R}+}, E_\mu)$, there is a positive σ-measure $\tilde{\mu}_f$ on B, uniquely determined by the properties:

.0. $\tilde{\mu}_f$ is a σ-measure,

.1. $\tilde{\mu}_f(a) = 0$ if $a \wedge \operatorname{supp} f = 0$ or $a \in N_\mu$,

and .2. if $a \in E_\mu$ and $0 \neq a \leq \operatorname{supp}(f)$, then (Mean Value)

$$m_f(a)\,\mu(a) \leq \tilde{\mu}_f(a) \leq M_f(a)\,\mu(a) .$$

In addition, every measure $\tilde{\mu}_f$ is almost μ-normal, and (Radon-Nikodym) every almost μ-normal positive measure on B is obtained in this way. $\tilde{\mu}_f = 0$ iff $\operatorname{supp}(f) \in N_\mu$. The function $\tilde{\mu}: (\mathbb{B}_{\mathbb{R}+}, E_\mu) \longrightarrow \left\{\begin{matrix}\text{almost } \mu\text{-normal} \\ \text{positive measures}\end{matrix}\right\}$ is additive and positive homogeneous, and satisfies the chain rule $\widetilde{(\tilde{\mu}_f)}_g = \tilde{\mu}_{(fg)} = \widetilde{(\tilde{\mu}_g)}_f$. Finally (Change of Variables) the passage from μ to $\tilde{\mu}$ is natural with respect to σ-morphisms: if $\phi \in \sigma(A, B)$ and $f \in \sigma(\mathbb{B}_{\mathbb{R}+}, E_{\mu\circ\phi})$, then $\widetilde{(\mu \cdot \phi)}_f = \tilde{\mu}_{(\phi \cdot f)} \cdot \phi$. Proof:

The proof must be preceded by a lemma.

(2.4.2) Lemma. If f is a bounded positive functionoid on E_μ with $\mathrm{supp}(f) \in F_\mu$, then the conditions

.o. ϕ is a σ-measure on B

.1. $\phi(a) = 0$ if $a \wedge \mathrm{supp}(f) = 0$ or $a \in N_\mu$

.2. $0 \neq a \leq \mathrm{supp}(f) \Longrightarrow m_f(a)\,\mu(a) \leq \phi(a) \leq M_f(a)\,\mu(a)$

are equivalent to the statement

.3. $\phi(a) = I_\mu(f\,\chi_a)$ for all $a \in B$.

Proof: If $\phi(a)$ is defined by .3., the facts that $\phi(a) = I_{\mu'}(f\,\chi_a)$, where $\mu'(b) = \mu(b \wedge \mathrm{supp}(f))$, and that $I_{\mu'}$ is Daniell (μ' being a finite σ-measure) establish .O.; .1. is obvious; and .2. is due to the fact that $I_{\mu'}$ preserves the inequalities

$$m_f(a)\,\chi_a \leq f\,\chi_a \leq M_f(a)\,\chi_a .$$

For the converse, suppose first that f is a step functionoid, $f = \sum_i r_i \chi_{a_i}$ (the a_i's pairwise disjoint). Then $m_f(a) = M_f(a) = r_i$ whenever $0 \neq a \leq a_i$, so that .O., .1., and .2. indicate that $\phi(a) = \sum_i \phi(a \wedge a_i) = \sum_i r_i\,\mu(a \wedge a_i) = I_\mu(f\,\chi_a)$ for all $a \in B$. Now, if f is arbitrary, let $\varepsilon > 0$ be provided, and choose an integer N so that, when $\delta = \frac{\|f\|_\infty}{N}$, $0 < \delta < \frac{\varepsilon}{2\,\mu(\mathrm{supp}(f))}$. For $k = 1, \ldots, N$, let

$$a_k = f((\delta(k-1), \delta k]), \quad f_k = f\,\chi_{a_k} .$$

Then $a_k = \mathrm{supp}(f_k)$ and if $0 \neq a \leq a_k$, $0 \leq M_{f_k}(a) - m_{f_k}(a) \leq \delta$. For any $b \leq \mathrm{supp}(f)$ $(b \neq 0)$, let $b_k = b \wedge a_k$. Then

$$0 \leq M_f(b_k) - m_f(b_k) = M_{f_k}(b_k) - m_{f_k}(b_k) \leq \delta \quad ,$$

whenever $b_k \neq 0$, and so by .2., $b_k \neq 0 \implies$

$$0 \leq \phi(b_k) - m_f(b_k)\,\mu(b_k) \leq \delta\mu(b_k) \;;$$

summing over k and using the fact that μ and ϕ are measures, we obtain

$$0 \leq \phi(b) - \sum_{k=1}^{N} m_f(b_k)\,\mu(b_k) = \sum_{k=1}^{N}(\phi(b_k) - m_f(b_k)\,\mu(b_k))$$

$$\leq \delta\sum_{k=1}^{N}\mu(b_k) = \delta\,\mu(b) < \frac{\varepsilon}{2} \;,$$

i.e.,

$$0 \leq \phi(b) - I_\mu(\chi_b \sum_{k=1}^{N} m_f(a_k)\,\chi_{a_k}) < \frac{\varepsilon}{2} .$$

On the other hand, the fact that

$$0 \leq f\,\chi_b - \chi_b \sum_{k=1}^{N} m_f(a_k)\,\chi_{a_k} \leq \delta\,\chi_b$$

implies that

$$0 \leq I_\mu(f\,\chi_b) - I_\mu(\chi_b \sum_{k=1}^{N} m_f(a_k)\,\chi_{a_k}) =$$

$$= I_\mu(f\,\chi_b - \chi_b \sum_{k=1}^{N} m_f(a_k)\,\chi_{a_k}) \leq$$

$$\leq I_\mu(\delta\,\chi_b) = \delta\,\mu(b) < \frac{\varepsilon}{2} \;,$$

and consequently $|\phi(b) - I_\mu(f\,\chi_b)| < \varepsilon$. Since ε was arbitrary, this proves the lamma.

To prove the uniqueness statement in the theorem, let f be any positive functionoid, and let $b \in B$. Where a_k $(k = 1, 2, \ldots)$ are any pairwise disjoint somas in F_μ whose union is $\mathrm{supp}(f)$, and $b_n = f((n-1, n])$ $(n = 1, 2, \ldots)$, define

$$a_{nk} = b_n \wedge a_k , \quad f_{nk} = f\chi_{a_{nk}} , \quad b_{nk} = b \wedge a_{nk} .$$

Since $\tilde{\mu}_f$ is supposedly a σ-measure vanishing on somas disjoint from $\operatorname{supp}(f)$, and since $\operatorname{supp}(f) = \bigvee_{n,k} a_{nk}$,

$$(*) \qquad \tilde{\mu}_f(b) = \tilde{\mu}_f(b \wedge \operatorname{supp}(f)) = \sum_{n,k} \tilde{\mu}_f(b_{nk}) .$$

Moreover, since $a_{nk} \in F_\mu$, we have $b_{nk} \in F_\mu$, and so, whenever $b_{nk} \neq 0$, the hypothesis in .2. is fulfilled and

$$m_{f_{nk}}(b_{nk})\,\mu(b_{nk}) = m_f(b_{nk})\,\mu(b_{nk}) \leq \tilde{\mu}_f(b_{nk}) \leq$$
$$\leq M_f(b_{nk})\,\mu(b_{nk}) = M_{f_{nk}}(b_{nk})\,\mu(b_{nk}) .$$

By Lemma (2.4.2), therefore, $\tilde{\mu}_f(b_{nk}) = I_\mu(f\chi_{b_{nk}})$; $\tilde{\mu}_f(b)$ is then determined by (*), and the proof of uniqueness is concluded.

We need another lemma to prove the existence statement.

(2.4.3) Lemma. If $(\phi_i)_{i \in I}$ is a family of positive σ-measures, if $i, j \in I \Longrightarrow \exists\, k \in I$ such that

$$\phi_i \leq \phi_k \geq \phi_j ,$$

and if the extended real valued function ϕ is defined by

$$\phi(a) = \sup_i \phi_i(a) ,$$

then ϕ is a σ-measure. Proof:

Suppose $a = \bigvee_{i=1}^{\infty} b_i$, where $b_i \wedge b_j = 0$ for $i \neq j$. Then, since $\phi_k(a) = \sum_i \phi_k(b_i) \leq \sum_i \phi(b_i)$ for each $k \in I$, we have

$$\phi(a) \leq \sum_i \phi(b_i) .$$

If $\phi(a)$ is finite (otherwise there's nothing to prove), let an integer n and a real number $\varepsilon > 0$ be given. Choose $k_i \in I$ so that $\phi_{k_i}(b_i) \geq \phi(b_i) - \frac{\varepsilon}{n}$ $(1 \leq i \leq n)$; then find $k \in I$ so that $\phi_{k_i} \leq \phi_k$ $(1 \leq i \leq n)$. We see

$$\sum_{i=1}^{n} \phi(b_i) \leq \sum_{i=1}^{n} (\phi_{k_i}(b_i) + \tfrac{\varepsilon}{n}) \leq \varepsilon + \sum_{i=1}^{n} \phi_k(b_i) \leq$$

$$\leq \varepsilon + \sum_{i=1}^{\infty} \phi_k(b_i) = \varepsilon + \phi_k(a) \leq \varepsilon + \phi(a) .$$

This being true for all ε and n, we see $\sum_i \phi(b_i) \leq \phi(a)$, which finishes the proof.

For the existence proof in Theorem (2.4.1), let f, a positive functionoid, be given. As in the uniqueness proof, find pairwise disjoint somas a_i in F_μ, whose union is $\text{supp}(f)$, for which $f_i = f\chi_{a_i}$ is bounded. Lemma (2.4.2) provides measures $\phi_i = \tilde{\mu}_{f_i}$: define $\tilde{\mu}_f = \sum_i \phi_i$ (the sup over all finite sums) -- this is a σ-measure by (2.4.3) and vanishes wherever each ϕ_i vanishes, so that it enjoys properties .1. and .2. of (2.4.1). To verify .3., suppose $0 \neq b \leq \text{supp}(f)$ has $\mu(b)$ finite. By (2.4.2) and (2.3.5),

$$m_f(b)\,\mu(a_i \wedge b) \leq m_f(a_i \wedge b)\,\mu(a_i \wedge b) = m_{f_i}(a_i \wedge b)\,\mu(a_i \wedge b)$$

$$\leq \phi_i(a_i \wedge b) \leq M_{f_i}(a_i \wedge b)\,\mu(a_i \wedge b) = M_f(a_i \wedge b)\,\mu(a_i \wedge b)$$

$$\leq M_f(b)\,\mu(a_i \wedge b) \qquad \text{(whenever } a_i \wedge b \neq 0\text{)};$$

since $\phi_i(a_i \wedge b) = \phi_i(b)$, summation over i yields

$$m_f(b)\,\mu(b) = m_f(b)\sum^* \mu(a_i \wedge b) \le \sum^* \phi_i(b) \le$$

$$\le \sum^* M_f(b)\,\mu(a_i \wedge b) = M_f(b)\,\mu(b)\,,$$

where $\sum^*$ means summation over those i for which $a_i \wedge b \neq 0$. Since $\tilde{\mu}_f(b) = \sum^* \phi_i(b)$ (because $\phi_i(b) = 0$ if $a_i \wedge b = 0$), this completes the existence proof.

Proof that $\tilde{\mu}_f$ is almost μ-normal. That $N_\mu \subseteq N_{\tilde{\mu}_f}$ and that $\mathrm{supp}(f) \in E_\mu$ is a carrier of $\tilde{\mu}_f$ are immediate. Next, let $b \in E_\mu$. Then $b \triangle (b \wedge \mathrm{supp}(f))$ is in $N_{\tilde{\mu}_f} \subseteq E_{\tilde{\mu}_f}$, and $b \wedge \mathrm{supp}(f)$ can be written as a countable union of disjoint somas $b_i \in F_\mu$ for which $f\chi_{b_i}$ is bounded; consequently $b_i \in F_{\tilde{\mu}_f}$, and so $b \in E_{\tilde{\mu}_f}$. Finally, we prove that $\tilde{\mu}_f$ and μ are comparable over $\mathrm{supp}(f)$. For each $t \in \mathbb{R}$, define

$$e_t^+ = f((0, t]),\quad e_t^- = f((t, +\infty))\,.$$

Then $e_t^+ \vee e_t^- = \mathrm{supp}(f)$, $e_t^+ \wedge e_t^- = 0$, and $0 \neq a \le e_t^+$ implies $\tilde{\mu}_f(a) \le M_f(a)\,\mu(a) \le M_f(e_t^+)\,\mu(a) \le t\,\mu(a)$, while $0 \neq a \le e_t^-$ implies $t\,\mu(a) \le m_f(e_t^-)\,\mu(a) \le m_f(a)\,\mu(a) \le \tilde{\mu}_f(a)$. (For $a = 0$ there is nothing to prove.)

Proof that every almost μ-normal positive measure is obtained. Let λ be one, and let (t, e_t^+, e_t^-) be a family of enough Hahn decompositions of a carrier e of λ. For each $r \in \mathbb{R}$ and a fixed countable set T, dense in $\mathbb{R}$, of parameters t among the available Hahn decompositions, define

$$S_r = \bigwedge \{e_t^+ / t \in T , t > r\} .$$

Observe that $(r, S_r, e \triangle S_r)_{r \in \mathbb{R}}$ is again a family of Hahn decompositions of e for λ with respect to μ, that

$$(*) \qquad \bigwedge_{s < r \in \mathbb{R}} S_r = S_s \quad \text{and} \quad S_s \leq S_r \ \text{ if } \ s \leq r ,$$

and that $S_0 \in N_\lambda$; to see the last relation, write S_0 as a union of countably many somas $S^i \in F_\mu$, notice that $\lambda(S^i) \leq t\,\mu(S^i)$ for each $0 < t \in T$ and each i, deduce that $S^i \in N_\lambda$, and conclude therefore that $S_0 \in N_\lambda$. Replacing e by $e \triangle S_0$ and each S_r by $S_r \triangle (S_r \wedge S_0)$ if necessary, we may assume $S_0 = 0$; i.e.,

$$(**) \qquad \bigwedge_{r > 0} S_r = 0 .$$

But it follows from $(*)$ and $(**)$ that there is a unique σ-morphism $f\colon \mathbb{B}_{\mathbb{R}^+} \longrightarrow B$ with $f((0, r]) = S_r$ (cf. Götz [11, §2.2]). Since $\operatorname{supp}(f) = \bigvee_r S_r \leq e$, $f \in \delta(\mathbb{B}_{\mathbb{R}^+}, E_\mu)$.

To see that $\tilde{\mu}_f = \lambda$, we check that λ satisfies the mean value theorem for f. From the definitions of m and M, we immediately obtain

$$m_f(a) = \sup \{r / S_r \wedge a = 0\} = \sup \{r / a \leq e \triangle S_r\}$$
$$M_f(a) = \inf \{r / a \leq S_r\}$$

whenever $0 \neq a \leq \operatorname{supp}(f)$; from these formulae, under the hypotheses of (2.4.1 .2.), we immediately obtain

$$\lambda(a) \geq m_f(a)\,\mu(a) , \quad \lambda(a) \leq M_f(a)\,\mu(a) ,$$

and so $\tilde{\mu}_f = \lambda$, by the uniqueness.

If $a \leq b$, then $a = b \wedge a$ and so $m_f(b)\chi_a = m_f(b)\chi_{b\wedge a}$ $= m_f(b)\chi_b\chi_a \leq f\chi_b\chi_a = f\chi_a$; hence $m_f(b) \leq m_f(a)$. Now, using (2.3.4), $M_f(a) = -m_{-f}(a) \leq -m_{-f}(b) = M_f(b)$. Finally, $m_f(a)\chi_a \leq f\chi_a \leq M_f\chi_a$ shows $m_f(a) \leq M_f(a)$, and completes the proof of (2.3.5).

To prove (2.3.6), observe that $r\chi_{b_i} \geq f\chi_{b_i}$ for all i if and only if $r\chi_b \geq f\chi_b$: this proves the second identity, and the first is now a consequence of (2.3.4).

A similar argument proves (2.3.7).

To rpove (2.3.8), notice first that the sup is $\leq M_f(b)$, according to (2.3.5). If the sup differs from $+\infty$ (otherwise we're through), call it S. We distinguish three cases.

i) $0 \neq b \leq \mathrm{supp}(f)$. Then b has non null intersection with some soma $f([n, n+1] - \{0\}) = a_n$ $(n \in Z)$ and $m_f(a_n \wedge b) \geq n$ for such a soma: hence $S \neq -\infty$. If $S \neq M_f(B)$, let $\varepsilon = M_f(b) - S > 0$, and for each integer n, define

$$b_n = f([M_f(b) - n^{-1}, M_f(b)] - \{0\}) \qquad (n \geq 1).$$

If $n > \frac{1}{\varepsilon}$, then $b_n \wedge b \neq 0$, and so $m_f(b \wedge b_n) \geq m_f(b_n) \geq$ $\geq M_f(b) - n^{-1}$, which proves the sup is $\geq M_f(b)$ and establishes the formula.

ii) $0 \neq b$, $b \wedge \mathrm{supp}(f) = 0$. Then $m_f(b) = 0 = M_f(b)$.

iii) $b = c \vee d$, $0 \neq c \leq \mathrm{supp}(f)$, $d \neq 0 = d \wedge \mathrm{supp}(f)$. This case follows from the previous cases by use of (2.3.6). This concludes the proof of (2.3.8).

(2.3.9) follows from (2.3.8) by use of (2.3.4).

(2.3.10, 11, and 12) all follow from the easily verified then $fg = \sum_i r_i s_i \chi_{a_i}$ and $m_{fg}(a_i) = r_i s_i = m_f(a_i) m_g(a_i)$; likewise for M, and it follows that $\lambda = \tilde{\mu}_{fg}$ in this case. It follows by an easy limit argument for the general case.

Naturality. Let $f \in \sigma(\mathbb{B}_{\mathbb{R}^+}, A)$, $\phi \in \sigma(A, B)$, and μ a σ-measure on B. Since $\mu \cdot \phi$ is then a σ-measure on A, and $\phi(e) \in E_\mu$ whenever $e \in E_{\mu\cdot\phi} \subseteq A$, it follows that $\phi \cdot f \in \sigma(\mathbb{B}_{\mathbb{R}^+}, E_\mu)$ whenever $f \in \sigma(\mathbb{B}_{\mathbb{R}^+}, E_{\mu\cdot\phi})$. Now $\widetilde{(\mu\cdot\phi)}_f$, as a σ-measure on A, is uniquely characterised by its mean value theorem. Let $a \in F_{\mu\cdot\phi}$ satisfy $0 \neq a \leq \mathrm{supp}(f)$. Then $\phi(a) \in F_\mu$, $\phi(a) \leq \mathrm{supp}(\phi\cdot f)$, and if $\phi(a) \neq 0$, the mean value theorem for $\tilde{\mu}_{\phi\cdot f}$ asserts

$$m_{\phi\cdot f}(\phi(a))\,\mu(\phi(a)) \leq \tilde{\mu}_{\phi\cdot f}(\phi(a)) \leq M_{\phi\cdot f}(\phi(a))\,\mu(\phi(a)) .$$

Now when $\phi(a) \neq 0$, $m_{\phi\cdot f}(\phi(a)) \geq m_f(a)$ and $M_{\phi\cdot f}(\phi(a)) \leq M_f(a)$; combining these inequalities with the mean value inequalities above for $\tilde{\mu}_{\phi\cdot f}(\phi(a))$ shows that $\tilde{\mu}_{\phi\cdot f} \cdot \phi$ enjoys the mean value inequalities appropriate to $\widetilde{(\mu\cdot\phi)}_f$, which proves they are the same.

The proof is ended, and some general remarks are called for.

1) It is customary to write

$$\tilde{\mu}_f(a) = \int_a f\,d\mu , \quad \tilde{\mu}_f(\mathrm{supp}(f)) = \int f\,d\mu .$$

A functionoid f is integrable (more precisely, μ-integrable) if (it is in $\sigma(\mathbb{B}_{\mathbb{R}^+}, E_\mu)$ and) $\int f\,d\mu$ is finite, which is the case if and only if $\tilde{\mu}_f$ is finite (in addition to being almost μ-normal), i.e., if and only if $\tilde{\mu}_f$ is μ-normal. If $f \in \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, E_\mu)$ and both f^+ and f^- are μ-integrable (equivalently, if $|f|$ is μ-integrable), f is <u>absolutely</u> μ-<u>integrable</u>, and we write $\tilde{\mu}_f = \tilde{\mu}_{f^+} - \tilde{\mu}_{f^-}$, $\int_a f\, d\mu = \tilde{\mu}_f(a)$, etc. From the additivity statement in the theorem and the description of the kernel of $\tilde{\mu}_f$, it follows that the μ-normal measures (viewed as a subspace of $W_1(B)$) and the quotient $L_1(\mu, B)$, of the space of absolutely μ-integrable functionoids in $\sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, E_\mu)$ by the subspace consisting of those whose support is in N_μ, are isomorphic as partially ordered linear spaces; this isomorphism is converted into an isometry by defining $\|f\|_1 = \|\tilde{\mu}_f\|_1$ for $f \in L_1(\mu, B)$.

2) Using the easily verifiable fact that, when $p: B \longrightarrow C$ is an $\underline{\underline{S}}$-epimorphism in $\sigma(B, C)$, each σ-measure on B, μ, for which $\ker(p) \subseteq N_\mu$, is of the form $\lambda \cdot p$ for a unique σ-measure λ on C, we show the three spaces

$$W_1(E_\mu/N_\mu),\quad L_1(\mu),\quad L_1(\mu, B)$$

are all canonically isomorphic with $L_1(\mu/N_\mu, B/N_\mu)$, where, if p is the canonical projection $B \longrightarrow B/N_\mu$, μ/N_μ is the unique σ-measure on B/N_μ whose composition with p is μ. From the naturality, we have an isometry of $L_1(\mu, B)$ into $L_1(\mu/N_\mu, B/N_\mu)$, and since $\mathbb{B}_{\mathbb{R}-\{0\}}$, being a retract of $\mathbb{B}_{\overline{\mathbb{R}}}$, is $\underline{\underline{S}}$-projective (cf. (1.5.0)), it follows that every absolutely integrable functionoid in

$L_1(\mu/N_\mu, B/N_\mu)$ is obtained. Next, we show that each finite σ-measure λ on E_μ/N_μ is in fact μ/N_μ-normal. (It might be pointed out, in this connection, that F_μ/N_μ and F_{μ/N_μ}, likewise E_μ/N_μ and E_{μ/N_μ}, are the same.) Indeed, each principal ideal of E_μ/N_μ is complete, for μ/N_μ is strictly positive on it, and so any disjoint family of non null somas can be at most countable. Consequently, if e is any soma outside of which λ, being bounded, vanishes, and if $t \in \mathbb{R}$, defining

$$e_t^+ = \bigvee \{a / a \le e\ ,\ \lambda(a) \le t(\mu/N_\mu)(a)\}\ ,\quad e_t^- = e \triangle e_t^+\ ,$$

(t, e_t^+, e_t^-) is a Hahn decomposition of e for λ with respect to μ/N_μ; over e, therefore, λ is comparable with μ/N_μ, and since λ vanishes on somas disjoint from e, λ is μ/N_μ-normal. This proves $W_1(E_\mu/N_\mu) \cong L_1(\mu/N_\mu, B/N_\mu)$. It is clear from the definition as a completion that $L_1(\mu)$ and $L_1(\mu/N_\mu)$ are isometrically isomorphic, and the inclusion of $\mathbb{R}\# F_{\mu/N_\mu}$ as step functionoids in $L_1(\mu/N_\mu, B/N_\mu) \subseteq \sigma(\mathbb{B}_{\mathbb{R}-\{0\}}, E_\mu/N_\mu)$ makes $\mathbb{R}\# F_{\mu/N_\mu}$ a dense, isometrically embedded subspace. The fact that $L_1(\mu/N_\mu, B/N_\mu)$ is complete in its norm, because W_1 is, guarantees that it is the completion of $\mathbb{R}\# F_{\mu/N}$, i.e., is $L_1(\mu/N_\mu)$, which completes the argument. If $L_p(\mu, B)$ denotes the classes, modulo null functionoids, of functionoids f for which $|f|^p$ is integrable ($|f|^p$ is definable in terms of the operation $|\ |^p$ on the absolute Borel space $(\mathbb{R}, \mathbb{B}_{\mathbb{R}}, 0)$) with the obvious norm, one can prove $L_p(\mu) \cong L_p(\mu, B)$, $1 \le p \in \mathbb{R}$.

3) To require that there be enough Hahn decompositions, in the definition of (almost) μ-normality is, as one of the arguments in 2) suggests, merely a technical convenience: any σ-measure satisfying the first three requirements of the definition is automatically comparable with μ over any carrier. That is an irrelevancy of sorts, in that it only makes the argument longer. Its proof, of course, is to divide out by N_μ , obtain Hahn decompositions in the quotient, and lift them up arbitrarily to the original σ-ring, taking care afterwrds to remove an appropriate soma of measure zero. One must work with only a countable number of decompositions in order to succeed in this program.

4) If f is a functionoid on a $\hat{\sigma}$-ring B , many would be tempted to write

$$f = \int x \, dE(x)$$

where $E(x) = f((-\infty, x])$, and define

$$\int f \, d\mu = \int_{\mathbb{R}} x \, d(\mu(E(x))),$$

the right hand integral being a Stieltjes integral over $\mathbb{R}$. Using x to denote also the identity map on $\mathbb{B}_{\mathbb{R}}$, we have

$$\int f \, d\mu = \int x \, d(\mu \cdot f)$$

by naturality, and the measure $\mu \cdot f$ is exactly the measure whose Radon-Nikodym derivative with respect to Haar measure is the function $\mu(E(\ ))$, so that this variant definition is compatible (everyone always knew it worked).

More generally, if g is a measurable function on the absolute Borel space $(\mathbb{R}, \mathbb{B}_{\mathbb{R}}, 0)$ and $\mathbb{B}(g)$ is the associated $\hat{\sigma}$-morphism from $\mathbb{B}_{\mathbb{R}}$ to itself, write $g[f] = f \cdot \mathbb{B}(g)$. By naturality, we obtain

$$\int g[f]\, d\mu = \int f \cdot \mathbb{B}(g)\, d\mu = \int \mathbb{B}(g)\, d(\mu \cdot f) =$$

$$= \int x\, d(\mu \cdot f \cdot \mathbb{B}(g)) = \int x\, d(\mu(f(g^{-1}((-\infty, x]))))$$

$$= \int g(x)\, d(\mu(f((-\infty, x]))) = \int g(x)\, d(\mu(E(x))),$$

where the last three integrals are Stieltjes type. This is familiar from the operational calculus of e.g. self-adjoint operators, where one writes $g[f] = \int g(x)\, dE(x)$.

5) There is no difficulty in extending the range of somas for which the mean value inequalities hold to all of B^0, for integrable functionoids as well as for positive functionoids in $\delta(\mathbb{B}_{\mathbb{R}^+}, E_\mu)$, provided, of course, that the product of 0 with $+\infty$ is interpreted as 0. The whole usefulness of these inequalities, however, is with somas that are small, but not zero, so far as the measure is concerned.

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$$\int g[f]\, d\mu = \int f\cdot\mathbb{B}(g)\, d\mu = \int \mathbb{B}(g)\, d(\mu\cdot f) =$$

$$= \int x\, d(\mu\cdot f\cdot\mathbb{B}(g)) = \int x\, d(\mu(f(g^{-1}((-\infty, x]))))$$

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2.5 The dual of L_1

Until further notice, assume that B is a complete boolean ring and that the σ-ideal E_B (cf. (2.2.1) and (2.2.3)) is dense in B, so that (by (1.10.9)) $B = \sigma E_B$. Define $J : \mathbb{R}\overline{\#} B \longrightarrow (L_1(B))^*$ by specifying $J(f)$, for $f \in \mathbb{R}\overline{\#} B$, to be the continuous linear functional on $L_1(B)$ given by the formula

$$(2.5.1) \qquad J(f)(\mu) = I_\mu(f) \qquad (\mu \in L_1(B)).$$

Observe that J is the same as the composition

$$\mathbb{R}\overline{\#} B \longrightarrow (\mathbb{R}\overline{\#} B)^{**} \cong (V_1(B))^* \longrightarrow (L_1(B))^*$$

of the canonical inclusion in the second dual, the transpose of the isomorphism of (2.1.7), and the transpose of the inclusion, hence is a norm-decreasing, order-preserving, continuous linear transformation.

(2.5.2) Theorem. J is an isometric isomorphism.

Proof: Let f be a step functionoid, say $f = \sum_i r_i \chi_{a_i} \neq 0$ with the a_i 's disjoint and $\neq 0$. Then $\|f\|_\infty = \max_i |r_i| = |r_1|$, say. Since $a_1 \neq 0$, there is a soma $e \in E_B$ for which $0 \neq e \leq a_1$; choose a strictly positive finite normal measure on the principal ideal B_e and extend it to a finite normal measure μ on B by defining it to be zero on somas disjoint from e. Then $\mu(a_1) = \mu(e) = \|\mu\|_1$ and $|J(f)(\mu)| = |I_\mu(f)| = |r_1 \mu(a_1)| = |r_1| \|\mu\|_1 = \|f\|_\infty \|\mu\|_1$, so that

$\|J(f)\| \geq \|f\|_\infty$. Since J is norm-decreasing, we have $\|J(f)\| = \|f\|_\infty$, and J is an isometry on the step functionoids. They are dense in $\mathbb{R}\overline{\#} B$, however, and so J is an isometry. It remains, then, to see that every linear functional on $L_1(B)$ is obtained, i.e., that for each $F \in (L_1(B))^*$ there is an $f \in \mathbb{R}\overline{\#} B$ with

$$(2.5.3) \qquad J(f) = F .$$

To this end, define continuous linear transformations

$$K : (L_1(B))^* \longrightarrow \mathbb{EN}(L_1(B), L_1(B))$$

$$L : \mathbb{R}\overline{\#} B \longrightarrow \mathbb{EN}(L_1(B), L_1(B))$$

$$H : \mathbb{EN}(L_1(B), L_1(B)) \longrightarrow (L_1(B))^*$$

by the formulae below, in which $F \in (L_1(B))^*$, $\mu \in L_1(B)$, $b \in B$, $f \in \mathbb{R}\overline{\#} B$, and $\Phi \in \mathbb{EN}(L_1(B), L_1(B)) = \{$bounded linear transformations from $L_1(B)$ to itself$\}$:

$$(2.5.4) \qquad ((K(F))(\mu))(b) = F(\mu_b) , \quad \text{where} \quad \mu_b(a) = \mu(b \wedge a)$$

$$(2.5.5) \qquad ((L(f))(\mu))(b) = I_\mu(f\chi_b) = \tilde{\mu}_f(b)$$

$$(2.5.6) \qquad (H(\Phi))(\mu) = (\Phi(\mu))(1) , \quad \text{where } 1 = \text{unit of } B .$$

The identities

$$(K(J(f))(\mu))(b) = J(f)(\mu_b) = I_{\mu_b}(f) = I_\mu(f\chi_b)$$

and

$$(H(K(F)))(\mu) = (K(F))(\mu)(1) = F(\mu_1) = F(\mu)$$

indicate that $KJ = L$ and $HK = id_{(L_1(B))^*}$, from which it follows that $J = HKJ = HL$. Consequently, if we are able, given $F \in (L_1(B))^*$, to produce an $f \in \mathbb{R}\#B$ for which

$$L(f) = K(F) , \tag{2.5.7}$$

we will have solved the problem posed by (2.5.3), i.e., proved that J is onto. We shall need a lemma which is related to the chain rule.

(2.5.8) Lemma. If $\Phi \in \underline{\mathrm{EN}}(L_1(B), L_1(B))$ satisfies $|\Phi(\lambda)|(b) = 0$ whenever $|\lambda|(b) = 0$ (all $b \in B$ and all $\lambda \in L_1(B)$), then $L(f)\cdot\Phi = \Phi\cdot L(f)$ for all $f \in \mathbb{R}\#B$.

Proof: Since L and Φ are linear and continuous, it is fortunately enough to prove that Φ and $L(\chi_a)$ commute for each characteristic functionoid χ_a. The hypothesis indicates that $|\Phi(L(\chi_a)(\lambda))|(b) = 0$ whenever $|L(\chi_a)(\lambda)|(b) = 0$, i.e., whenever $a \wedge b \in N_{|\lambda|}$, and in particular, whenever $a\wedge b = 0$. Consequently, where $\mathbb{1} = \chi_1$ is the unit element of $\mathbb{R}\#B$, we have

$$\Phi(L(\chi_a)(\lambda))((1\triangle a)\wedge b) = 0$$

and

$$\Phi(L(\mathbb{1}-\chi_a)(\lambda))(a\wedge b) = 0 .$$

Hence

$$\Phi(\lambda)(b) = (\Phi\cdot(L(\chi_a)+L(\mathbb{1}-\chi_a))(\lambda))(b) =$$

$$= (\Phi\cdot L(\chi_a)(\lambda))(b) + (\Phi\cdot L(\mathbb{1}-\chi_a)(\lambda))(b) =$$

$$= \left\{\begin{matrix}(\Phi\cdot L(\chi_a)(\lambda))(a\wedge b)\\ +\\ (\Phi\cdot L(\chi_a)(\lambda))((1\triangle a)\wedge b)\end{matrix}\right\} + \left\{\begin{matrix}(\Phi\cdot L(\mathbb{1}-\chi_a)(\lambda))(a\wedge b)\\ +\\ (\Phi\cdot L(\mathbb{1}-\chi_a)(\lambda))((1\triangle a)\wedge b)\end{matrix}\right\} =$$

$$= (\Phi\cdot L(\chi_a)(\lambda))(a\wedge b) + (\Phi\cdot L(\mathbb{1}-\chi_a)(\lambda))((1\triangle a)\wedge b),$$

and by (2.5.5) this is the same as

$$(L(\chi_a)\cdot\Phi\cdot L(\chi_a)(\lambda))(b) + (L(\mathbb{1}-\chi_a)\cdot\Phi\cdot L(\mathbb{1}-\chi_a)(\lambda))(b).$$

Thus $\Phi = L(\chi_a)\Phi L(\chi_a) + L(\mathbb{1}-\chi_a)\Phi L(\mathbb{1}-\chi_a)$. Pre- and post-multiplying this identity by $L(\chi_a)$, we obtain

$$L(\chi_a)\Phi = L(\chi_a)\Phi L(\chi_a) = \Phi L(\chi_a) ,$$

as required.

Returning to the proof of Theorem (2.5.2), we are given $F \in L_1(B)^*$ and search for $f \in \mathbb{R}\overline{\#}B$ satisfying (2.5.7). Assume for the moment that $E_B = B$, so that there is a strictly positive finite normal measure μ defined on all of B . For convenience in thinking, assume also that F is a positive linear functional -- this is no great restriction. Notice that $K(F)(\mu)$ is then positive, and that, from the definition of K , $|K(F)(\lambda)|(b) = 0$ if $|\lambda|(b) = 0$. Then by (2.5.8), $K(F)\cdot L(g) = L(g)\cdot K(F)$ for all $g \in \mathbb{R}\overline{\#}B$. Using the Radon-Nikodym theorem, find an absolutely μ-integrable functionoid f for which $K(F)(\mu) = \tilde{\mu}_f$; since $0 \leq K(F)(\mu)(b) \leq \|F\|\,\mu(b)$, f is bounded, i.e., is in $\mathbb{R}\overline{\#}B$, and $L(f)(\mu)(b) = I_\mu(f\chi_b) = \tilde{\mu}_f(b) = K(F)(\mu)(b)$, so that

$$L(f)(\mu) = K(F)(\mu) . \tag{2.5.9}$$

This is not quite good enough: we need to know that $L(f)(\lambda) = K(F)(\lambda)$ for every finite normal measure λ on B . So let $\lambda \in L_1(B)$. Using the Radon-Nikodym theorem once again, produce an absolutely μ-integrable functionoid g such that $\lambda = \tilde{\mu}_g$. If $g \in \mathbb{R}\overline{\#} B$ (which of course need not be the case), then $\lambda = L(g)(\mu)$, hence

$$\begin{aligned} L(f)(\lambda) &= L(f)(L(g)(\mu)) = L(g)(L(f)(\mu)) = \\ &= L(g)(K(F)(\mu)) = K(F)(L(g)(\mu)) = \\ &= K(F)(\lambda), \end{aligned}$$

using only Lemma (2.5.8). That measures $\tilde{\mu}_g$ with $g \in \mathbb{R}\overline{\#} B$ are dense in $L_1(B)$ follows from the second remark after (2.4.1), and so by continuity, $L(f)(\lambda) = K(F)(\lambda)$ for all $\lambda \in L_1(B)$. This establishes (2.5.7) in the case that $E_B = B$, and proves the theorem in that case.

For the general case, that E_B is merely dense in B, notice that every finite normal measure on B is the extension by zero of a finite normal measure on a principal ideal B_e, $e \in E_B$ (compare Lemmas (2.2.4) and (2.2.5)). In this sense,

$$L_1(B) = \bigcup_{e \in E_B} L_1(B_e) = \operatorname*{dir\,lim}_{e \in E_B} L_1(B_e) ,$$

so that

$$\begin{aligned} (L_1(B))^* &= (\operatorname*{dir\,lim}_{e \in E_B} L_1(B_e))^* = \operatorname*{inv\,lim}_{e \in E_B} (L_1(B_e))^* = \\ &= \operatorname*{inv\,lim}_{e \in E_B} \mathbb{R}\overline{\#} B_e = \mathbb{R}\overline{\#}(\operatorname*{inv\,lim}_{e \in E_B} B_e) = \\ &= \mathbb{R}\overline{\#}\, \beta E_B = \mathbb{R}\overline{\#} B , \end{aligned}$$

which proves the theorem (using the results of §1.10 and the already established special case).

(2.5.10) Corollary. If μ is a positive σ-measure on a σ-ring B, then $(L_1(\mu))^* = \mathbb{R}\overline{\#}(\beta(E_\mu/N_\mu))$.

Proof: By the remarks after (2.4.1) and by (2.2.4) and (2.2.5), we have isomorphisms

$$L_1(\mu) = W_1(E_\mu/N_\mu) = L_1(\beta(E_\mu/N_\mu)).$$

Now apply (2.5.2).

(2.5.11) Corollary (Thorp [30, Theorem 4]). Let μ be a positive σ-measure on a $\hat{\sigma}$-ring B. Assume that $B = \beta E_\mu$ and that $N_\mu = \tilde{N}_\mu$. Then $(L_1(\mu))^* = \mathbb{R}\overline{\#}(B/N_\mu)$. Proof:

Immediate consequence of the previous corollary and the definitions of §2.3.

False. See §2.7.

Remarks. 1) It follows from (2.2.4), (2.2.5), and (2.5.2) that for an arbitrary complete boolean ring, B, $(L_1(B))^* = \mathbb{R}\overline{\#}\beta E_B$.

2) The passage from the complete boolean ring B to its σ-ideal and $\wedge$-complete boolean ring E_B is the construction, promised in the remark after the proof of (1.10.11), that behaves, at least occasionally, like an inverse to the functor $\beta : \mathcal{X} \longrightarrow \mathcal{Y}$. We know no more about it than is presented in this chapter.

3) Combining the results of the present section with those of §2.2, we see that the canonical injection of $L_1(\mu)$ into its second dual $(L_1(\mu))^{**} = V_1(\beta(E_\mu/N_\mu))$

has a preferred splitting. This phenomenon does not seem to have been made use of. It suggests, speaking loosely, that $L_1(\mu)$ is trying to be reflexive. This is suggested also by the following observation. Call a continuous linear transformation between conditionally complete lattice-ordered Banach spaces <u>utterly continuous</u> if its positive and negative parts preserve the limits of all order-convergent monotone nets. Continuity and utter continuity are the same for linear functionals on $L_p(\mu)$ $(1 \leq p \in \mathbb{R})$, but a linear functional on $L_\infty(\mu) = \mathbb{R}\#\beta(E_\mu/N_\mu)$ is utterly continuous if and only if it is completely additive and Daniell. Thus, in the category of conditionally complete lattice-ordered Banach spaces and utterly continuous maps, $L_1(\mu)$ <u>is</u> reflexive.

4) As expected, it can be proved that $L_p(\mu)$ is reflexive, $1 < p \in \mathbb{R}$, with dual space $L_q(\mu)$ where $p^{-1} + q^{-1} = 1$; there is no great interest in this result, however, since it follows (through the Stone space) from the classical L_p-L_q duality.

2.6 The Fubini Theorem

D. A. Kappos seems to be the first to have formulated a Fubini theorem entirely in the context of boolean rings [18]. Unfortunately, he did not succeed in proving it, as was noticed by Sikorski [24] not long afterward. In the meantime, the matter appears to have been forgotten, and no purely boolean proof of the Fubini theorem has appeared, to the best of our knowledge, until the present.

(2.6.1) Theorem (Fubini). Let μ_1 and μ_2 be positive σ-measures on σ-rings B_1 and B_2, respectively. There is a unique σ-measure $\mu = \mu_1 \otimes \mu_2$ on the tensor product $E_{\mu_1} \otimes_{\sigma} E_{\mu_2}$ satisfying the equation

$$\mu(a \otimes b) = \mu_1(a)\ \mu_2(b) \qquad (a \in E_{\mu_1},\ b \in E_{\mu_2})$$

(the convention is $0 \cdot \infty = 0$). In addition, there is a canonical isometric isomorphism

$$L_1(\mu_1) \hat{\otimes}_{\gamma} L_1(\mu_2) \cong L_1(\mu_1 \otimes \mu_2)\ ,$$

where $\hat{\otimes}_{\gamma}$ denotes the tensor product (cf. Schatten [22] or Grothendieck [12]) in the autonomous category of Banach spaces. Proof:

Uniqueness follows from [2, §206, Satz 1]. For the existence, it is convenient and, by (2.4.1), harmless so far as the spaces L_1 and the measures are concerned, to assume that $N_{\mu_i} = \{0\}$ and $B_i = E_{\mu_i}$ (divide out N_{μ_i} and restrict attention ot E_{μ_i}, if necessary), since tensoring is exact.

Let $$i_{\mu_k} : \mathbb{R}\# F_{\mu_k} \longrightarrow \mathbb{R} \qquad (k = 1, 2)$$

be the positive linear functionals in terms of which the L_1 norms on $\mathbb{R}\# F_{\mu_k}$ are defined. Their tensor product

$$i_{\mu_1} \otimes_{\mathbb{R}} i_{\mu_2} : \mathbb{R}\# F_{\mu_1} \otimes_{\mathbb{R}} \mathbb{R}\# F_{\mu_2} \longrightarrow \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{R} ,$$

when composed with the canonical isomorphism of (1.6.7 .6.)

$$\mathbb{R}\# (F_{\mu_1} \otimes_2 F_{\mu_2}) \longrightarrow \mathbb{R}\# F_{\mu_1} \otimes_{\mathbb{R}} \mathbb{R}\# F_{\mu_2} ,$$

gives a positive linear functional i_* on $\mathbb{R}\# (F_{\mu_1} \otimes_2 F_{\mu_2})$. The norm $\| \|_{1,*}$ on $\mathbb{R}\# (F_{\mu_1} \otimes_2 F_{\mu_2})$,

$$\|f\|_{1,*} = i_*(f^+) + i_*(f^-) ,$$

associated to i_* is the same as the restriction to $\mathbb{R}\# F_{\mu_1} \otimes_{\mathbb{R}} \mathbb{R}\# F_{\mu_2}$ of the γ-crossnorm of Schatten on $L_1(\mu_1) \hat{\otimes}_\gamma L_1(\mu_2)$, and since $\mathbb{R}\# F_{\mu_k}$ is dense in $L_1(\mu_k)$ by definition, it follows that $\mathbb{R}\# (F_{\mu_1} \otimes_2 F_{\mu_2})$ is an isometrically embedded dense subspace of $L_1(\mu_1) \hat{\otimes}_\gamma L_1(\mu_2)$. Let F_o denote the topological closure in $L_1(\mu_1) \hat{\otimes}_\gamma L_1(\mu_2)$ of the set F of characteristic functionoids in

$\mathbb{R}\# (F_{\mu_1} \otimes_2 F_{\mu_2})$. Properties of the norm and order indicate that F_o is a $\bigwedge$-complete boolean ring and that μ_o, the restriction of the γ-crossnorm to F_o, is a strictly positive measure which is normal on each principal ideal. Define a ζ-morphism $q: F_{\mu_1} \otimes_\delta F_{\mu_2} \longrightarrow F_o$ by specifying the

$\mathcal{S}$-bilinear map $\tilde{q}$ that gives rise to q by the formula

$$\tilde{q}(a, b) = \chi_a \otimes \chi_b \qquad (a \in F_{\mu_1}, b \in F_{\mu_2}).$$

This map q may not be onto, but, where $\sigma : \mathcal{S} \longrightarrow \mathcal{s}$ is the left adjoint of the inclusion functor $\mathcal{s} \longrightarrow \mathcal{S}$,

$$\sigma(q): \sigma(F_{\mu_1} \otimes_{\mathcal{S}} F_{\mu_2}) \longrightarrow \sigma(F_o)$$

is onto. Indeed, it's enough to see that each element of F_o is obtained. But each element of F_o is the norm limit of a sequence of elements of F, which are in the image, hence is obtained. Now the measure μ_o on F_o, being normal on each principal ideal, has a unique extension to a $\mathcal{S}$-measure on $\sigma(F_o)$, say $\sigma(\mu_o)$, and it is evident that $F_o = F_{\sigma(\mu_o)}$. Hence

$$L_1(\sigma(\mu_o)) = L_1(\mu_o) = L_1(\mu_1) \hat{\otimes}_{\gamma} L_1(\mu_2).$$

Finally, $\sigma(F_{\mu_1} \otimes_{\mathcal{S}} F_{\mu_2}) = \sigma(F_{\mu_1}) \otimes_{\sigma} \sigma(F_{\mu_2}) = E_{\mu_1} \otimes_{\sigma} E_{\mu_2}$ (this can be proved by arguments similar to the transfinite induction argument in the proof of (1.4.6) and (1.4.8)); hence, pulling the measure $\sigma(\mu_o)$ back over $\sigma(q)$ and these isomorphisms, we obtain a measure μ on $E_{\mu_1} \otimes_{\mathcal{S}} E_{\mu_2}$ for which $L_1(\mu) = L_1(\sigma(\mu_o)) = L_1(\mu_1) \hat{\otimes}_{\gamma} L_1(\mu_2)$ by the naturality statement in (2.4.1). From this relation and the definition of q, the desired product formula $\mu(a \otimes b) = \mu_1(a)\mu_2(b)$ is evident, and the theorem is proved.

Remarks. 1) The usual double integral formulation of the Fubini theorem is now trivial. If μ_1 and μ_2 are σ-measures and $\mu = \mu_1 \otimes \mu_2$ is the product measure as defined in the preceding theorem, the isomorphism $L_1(\mu_1) \hat{\otimes}_{\gamma} L_1(\mu_2) = L_1(\mu)$ indicates that the linear functional "integrate with μ" is the composition

$$L_1(\mu) = L_1(\mu_k) \hat{\otimes}_{\gamma} L_1(\mu_{3-k}) \longrightarrow L_1(\mu_{3-k}) \longrightarrow \mathbb{R}$$

where the second map is "(integrate with μ_k) $\otimes$ (ident.)" and the last is "integrate with μ_{3-k}" $(k = 1, 2)$.

2) $L_p(\mu_1) \otimes_{\mathbb{R}} L_p(\mu_2)$ can be proved to be dense in $L_p(\mu_1 \otimes \mu_2)$ $(1 \leq p \in \mathbb{R})$, but no L_p-type tensor product of Banach spaces appears to be known, else there would undoubtedly be an isomorphism. Schatten [23] gives a simple example showing that no such isomorphism is to be expected for L_∞. However, it is possible that working in the category of conditionally complete lattice-ordered Banach spaces with utterly continuous maps, there might be room for hope, since $L_\infty(\mu_1 \otimes \mu_2)$ does seem to be the only conditionally complete vector sublattice of itself containing the elementary tensor products $\chi_a \otimes \chi_b$.

This § has been added later. References to §1.10 refer to the revised §1.10, and not to the original.

2.7 Localizability and the dual of L_1

This section is divided into three parts. The first, culminating in (2.7.9), discusses the localizability of a boolean ring over one of its ideals, relating this notion to the material of §1.10. The second part uses localizability as a necessary and sufficient condition for a certain alternate description of the dual of L_1 to be valid. The third part relates localizability to the notion of localizability developed by Segal[(1)] in his study of measure spaces.

I. <u>Localizability over ideals</u>. Let J be an ideal in the boolean ring A, and let $p: A \longrightarrow A/J$ denote the natural projection. Thinking of A as an A-module, J becomes an A-submodule, and so the quotient ring A/J also inherits an A-module structure, which, incidentally, is the same as that induced from the natural (A/J)-module structure by change of rings via the canonical projection p. Write $d_{A,J}$ for the connecting homomorphism ${}_{A}\underline{\underline{M}}(A, A/J) \longrightarrow \mathrm{Ext}^1_A(A, J)$. Call A <u>localizable over</u> J if $d_{A,J} = 0$, so that $d_{A,J}$ is in a certain sense the obstruction to the localizability of A over J. The exactness of the initial portion

$$(2.7.1) \qquad 0 \to {}_{A}\underline{\underline{M}}(A, J) \to {}_{A}\underline{\underline{M}}(A, A) \to {}_{A}\underline{\underline{M}}(A, A/J) \to \mathrm{Ext}^1_A(A, J)$$

of the ext sequence immediately delivers

(2.7.2) <u>Observation</u>. A is localizable over J if and only if there is, for each A-module morphism $f: A \longrightarrow A/J$, an A-module endomorphism g of A such that $f = p \cdot g$.

Recalling (1.10.10), we now reinterpret

$$(2.7.3) \qquad {}_{A}\underline{\underline{M}}(A, p): {}_{A}\underline{\underline{M}}(A, A) \longrightarrow {}_{A}\underline{\underline{M}}(A, A/J)$$

as a map from βA to $\beta(A/J)$. Precisely, the projection $p: A \longrightarrow A/J$ induces maps

$$(2.7.4) \qquad {}_{A/J}\underline{\underline{M}}(A/J,\ A/J) \rightarrow {}_{A}\underline{\underline{M}}(A/J,\ A/J) \rightarrow {}_{A}\underline{\underline{M}}(A,\ A/J)$$

by change of rings and contravariant composition, respectively. Both maps are isomorphisms. For the first, this is obvious because the A-module structure on A/J is precisely that induced through p by change of rings. As for the second, which is certainly a monomorphism, it suffices to see that each $f \in {}_{A}\underline{\underline{M}}(A,\ A/J)$ vanishes on J; and, in fact, if $j \in J$,

$$f(j) = f(j \wedge j) = f(j) \wedge p(j) = f(j) \wedge 0 = 0.$$

Combining (2.7.3) with the isomorphisms afforded by (2.7.4) and (1.10.10 i), we obtain a map

$$\beta p:\ \beta A \longrightarrow \beta(A/J).$$

It is left for the reader to satisfy himself that βp is in fact a ring homomorphism, sending unit to unit.

(2.7.5) Lemma. βp is surjective if and only if A is localizable over J.

Proof: an immediate consequence of (2.7.2) and the fact that βp is surjective iff (2.7.3) is.

(2.7.6) Corollary. Every boolean ring with unit is localizable over every ideal (since $p = \beta p$).

Because of the exactness of (2.7.1), it is easy to characterise the kernel $\tilde{J}$ of βp.

(2.7.7) Lemma. The kernel $\tilde{J}$ of βp consists of those $b \in \beta A$ for which $a \in A \Longrightarrow b \wedge a \in J$.

The proof resides in the realisation that the described elements constitute ${}_{A}\underline{\underline{M}}(A,\ J)$.

Combining (2.7.7) with (2.7.5) for future reference, we have

(2.7.8) Proposition. The canonical map $\beta p: \beta A \longrightarrow \beta(A/J)$ induces a monomorphism $\tilde{p}: \beta A/\tilde{J} \longrightarrow \beta(A/J)$ which is an isomorphism if and only if A is localizable over J.

The most useful result for the rest of this section is

(2.7.9) Proposition. Let J be an ideal in A, and assume that $\beta(A/J)$ is complete (necessary and sufficient conditions for this to occur are given in (1.10.11)). The following conditions are then equivalent:

i) A is localizable over J; ii) p is surjective; iii) $\beta A/\tilde{J}$ is complete; iv) $\tilde{p}$ is an isomorphism.

Proof: i), ii), and iv) are mutually equivalent by (2.7.5) and (2.7.8); that iv) $\Longrightarrow$ iii) is trivial. It remains to prove, say, iii) $\Longrightarrow$ iv). Using (2.7.7), one sees that the kernel of the composition $A \longrightarrow \beta A \longrightarrow \beta A/\tilde{J}$ is precisely J. The induced map $A/J \longrightarrow \beta A/\tilde{J}$ is part of a tower of monomorphisms $A/J \longrightarrow \beta A/\tilde{J} \longrightarrow \beta(A/J)$, hence is a dense extension of A/J, and is therefore maximal, by (1.10.13). It follows, using (1.10.3), that $\tilde{p}$ is an isomorphism.

II. Measures on σ-rings. Let μ be a positive σ-measure on the σ-ring B, and let N_μ, F_μ, E_μ, B_μ have the same meaning as in §2.3 (we recall that $B_\mu = \beta E_\mu \cong \beta F_\mu$ and that $\beta(E_\mu/N_\mu) \cong \beta(F_\mu/N_\mu)$). Letting p denote either of the canonical projections $E_\mu \longrightarrow E_\mu/N_\mu$ or $F_\mu \longrightarrow F_\mu/N_\mu$, we get the same map $\beta p: B_\mu \longrightarrow \beta(E_\mu/N_\mu) \cong \beta(F_\mu/N_\mu)$, so that $\tilde{p}$ and $\tilde{N}_\mu$ are unambiguously defined. Say that the system (μ, B) is localizable if E_μ (equivalently, F_μ)

is localizable over N_μ.

(2.7.10) <u>Theorem</u>. The system (μ, B) is localizable if and only if either of the equivalent conditions listed below is valid.

i) The linear transformation $j: \mathbb{R} \# B_\mu \longrightarrow (L_1(\mu, B))^*$ given by

(2.7.11) $\quad j(f)(g) = \int fg\, d\mu \quad (f \in \mathbb{R} \# B_\mu,\ g \in L_1(\mu, B))$

induces an isometric isomorphism between $\mathbb{R} \# B_\mu / \mathbb{R} \# \tilde{N}_\mu$ and $(L_1(\mu, B))^*$;

ii) $B_\mu / \tilde{N}_\mu$ is complete.

<u>Proof</u>: The equivalence of ii) with either definition of localizability of (μ, B) is due to (2.7.9). For the rest of the proof, we present a factorisation of j:

$$\mathbb{R} \# B_\mu \longrightarrow \frac{\mathbb{R} \# B_\mu}{\mathbb{R} \# \tilde{N}_\mu} \xrightarrow{\tilde{=}} \mathbb{R} \# (B_\mu / \tilde{N}_\mu) \longrightarrow \mathbb{R} \# \beta(E_\mu / N_\mu) \xrightarrow{\simeq} (L_1(\mu, B))^* .$$

Here the first map is the canonical projection, the second is the canonical isomorphism, the third is $\mathbb{R} \# \tilde{p}$, and the last is the isomorphism of (2.5.10). By (2.7.9), to prove i) $\Leftrightarrow$ ii), it suffices to porve that $\mathbb{R} \# \tilde{p}$ is an isometric isomorphism if and only if $\tilde{p}$ is an isomorphism. That, however, is obvious.

III. <u>Measure spaces à la Segal</u>. The terminology to be introduced here is borrowed from the work of Segal cited in footnote (1). The only difference is that, while Segal allows complex valued functions, we restrict ourselves to real valued functions. This is no great restriction, however, and is done entirely for the sake of convenience. (Tensor everything with the complex numbers, if you want the complex case.)

A _measure space_ is a triple (X, R, m) consisting of a set X, a δ-ring R of subsets of X, and a finite positive measure m on R, whose restriction to each principal ideal of R is countably additive. We shall never make use of the further restriction that Segal imposes _per definitionem_ on all measure spaces, namely, that the union of each countable family of mutually disjoint sets $E_n \in R$ $(n = 1, 2, \ldots)$, for which $\sum_n m(E_n) < \infty$, also be in R.

A subset Y of X is _measurable_ if each intersection $Y \cap r$ $(r \in R)$ belongs to R. Segal verifies that the class B_1 of measurable sets is a σ-ring, that the function μ_1, defined by $\mu_1(Y) = \sup_{r \in R} m(Y \cap r)$, is a σ-measure on B_1, and that $\mu_1(r) = m(r)$ whenever $r \in R$. A real valued function on X is _measurable_ if, in the terminology of §1.7, it is a _Borel_-morphism from (X, R, θ) to $(\mathbb{R}, \mathbb{B}_{\mathbb{R}}, 0)$. A subset Y of X is _null_ if it is measurable and $m(Y \cap r) = 0$, whenever $r \in R$ (equivalently, $\mu_1(Y) = 0$); a function is _null_ if it is measurable and its support is a null set. A function is _essentially bounded_ if its restriction to the complement of some null set is bounded; _absolutely integrable_ if it is measurable and its absolute value is the pointwise limit of an L_1-Cauchy sequence of finite linear combinations of characteristic functions of sets in R. The space of essentially bounded measurable functions modulo null functions is denoted $RB_\infty(X, R, m)$, and has the sup norm; that of absolutely integrable functions mod null functions, in the L_1 norm, is $RB_1(X, R, m)$. We write RB_∞, RB_1, for short.

Letting N_1 denote the class of null sets, Segal calls (X, R, m) _localizable_ if the quotient ring B_1/N_1 is complete.

Segal proves[2] that the map $j: RB_{\infty} \longrightarrow (RB_1)^*$ defined by analogy with formula (2.7.11) is an isometry iff (X, R, m) is localizable. Corollary (2.5.10), however, provides an explicit, function-like representation of $(RB_1)^*$ whether (X, R, m) is localizable or not. Precisely, Theorems (1.7.1) and (1.7.2) permit complete identification of the space of measurable functions with the space $\partial(\mathbb{B}_{\mathbb{R}}, B_1)$, so that, using the projectivity of $\mathbb{B}_{\mathbb{R}}$, complete identification of the space of measurable functions mod null functions with $\partial(\mathbb{B}_{\mathbb{R}}, B_1/N_1)$ is possible. In this way, $RB_{\infty} \cong \mathbb{R}\#(B_1/N_1)$. Before we find the counterpart to RB_1, let us examine B_1 and N_1.

Let X_R denote the union (in X) of all the sets belonging to R, and write X_R' for its complement.

(2.7.12) Each subset of X_R' is measurable and null. If B and N denote the principal ideals of B_1 and N_1, respectively, generated by X_R, while $J = R \wedge N$, then $B = \beta R$, $N = \tilde{J}$, and $B_1/N_1 \cong B/N \cong \beta R/\tilde{J}$, where $\tilde{J}$ is the kernel of the canonical map $\beta R \longrightarrow \beta(R/J)$.

<u>Proof</u>: The first assertion is obvious. That $B = \beta R$ is due to (1.10.10) and the fact that $B = \beta_X(R)$ by construction. The next identification is due to (2.7.7), and the rest follows.

Returning to RB_1, let μ denote the restriction of μ_1 to B, and form N_μ, F_μ, E_μ, and B_μ as usual for the system (μ, B). Note that $N = N_\mu$ and that the image in B/N of E_μ is precisely the σ-ring generated by the image of R (under Segal's extra condition, which we are not using, the images of F_μ and of R would coincide).

Now RB_1 is defined essentially as the L_1-norm completion of $\mathbb{R}\#(R/J)$; $L_1(\mu, B)$, on the other hand, is the L_1-norm completion of $\mathbb{R}\#(F_\mu/N_\mu)$; what we have just said, however, ensures that each element of $\mathbb{R}\#(F_\mu/N_\mu)$ is approximable by elements of $\mathbb{R}\#(R/J)$, and so these completions coincide, i.e., for RB_1 we may take $L_1(\mu, B)$. We now prove

(2.7.13) <u>Theorem</u>. The following three conditions on the measure space (X, R, m) are equivalent.

i) (X, R, m) is localizable in the sense of Segal;

ii) R is localizable over J ;

iii) the map $j: RB_\infty \longrightarrow (RB_1)^*$ defined by analogy with the map (2.7.11) is an isometric isomorphism.

<u>Proof</u>: The equivalence of i) with ii) is due to (2.7.9) and (2.7.12). For the rest, let σ denote the adjoint to the inclusion of σ-rings in δ-rings. By some of the comments preceding the theorem, we have a tower

$$R/J \subseteq \sigma R/(N \cap \sigma R) = \sigma(R/J) = E_\mu/N \subseteq B/N = \beta R/\tilde{J} \subseteq \beta(R/J) .$$

Now E_μ/N is a dense ideal in B/N since E_μ is an ideal in B and the subset R/J is already dense in the over-ring $\beta(R/J)$. An application of (1.10.13) shows that the inclusion of E_μ/N in $\beta R/\tilde{J}$ makes $\beta R/\tilde{J} = \beta(E_\mu/N)$, whenever i) holds; on the other hand, i) is valid as sonn as $\beta R/\tilde{J} = \beta(E_\mu/N)$, since the latter is complete. To clinch the argument, one proceeds, as in the proof of (2.7.10), to show that iii) is the case also if and only if the inclusion of E_μ/N in $\beta R/\tilde{J}$ makes $\beta R/\tilde{J} = \beta(E_\mu/N)$, using the facts that $RB_\infty = \mathbb{R}\#(\beta R/\tilde{J})$ and $(RB_1)^* = (L_1(\mu, B))^* = \mathbb{R}\#\beta(E_\mu/N)$.

(1) I. E. Segal, Equivalences of measure spaces, Amer. J. Math. 73 (1951), pp. 275-313.

(2) Part of Theorem 5.1 on page 301 of the cited work of Segal.

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